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Mean curvature flow on graphs for image and manifold restoration and enhancement

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ABSTRACT

In this paper, we propose an adaptation and a transcription of the mean curvature level set equation on the general discrete domain, a weighted graph. For this, we introduce perimeters on graphs using difference operators and define the curvature as the first variation of these perimeters. Then we propose a morphological scheme that unifies both local and nonlocal notions of mean curvature on Euclidean domains. Furthermore, this scheme allows to extend the mean curvature applications to process images, manifolds and data which can be represented by graphs.

1. Introduction

In this paper, we present an adaptation of mean curvature flow level set equation on weighted graphs using the framework of Partial difference equation [7,9]. This adaptation aims to extend the mean curvature equation applications to any discrete data that can be represented by graphs. Moreover, it leads to a finite difference equation with data depending on coefficients whose solution gives rise to a new class of morphological operators for data restoration and enhancement.

1.1. Context and motivation

With the advent of our digital world, many different kinds of data are now available (images, meshes, social networks, etc.) that do not necessarily lie on a Cartesian grid and that can be irregularly distributed. To represent these data, the most natural and flexible representation consists in using weighted graphs by modeling neighborhood relationships. Processing these data on graphs is then a major challenge for image processing and machine learning communities, to address many applications, such as denoising, enhancement and clustering.

Historically, the main tools for the study of graphs or networks come from combinatorial and graph theory. Recently, there has been increasing interest in the investigation of two major mathematical tools for signal and image analysis, which are PDEs and wavelet on graph [8]. In particular, the PDE on graph was used in different applications that include filtering, denoising, segmentation and clustering, see [7,9,12,16–20] and references therein for more details. In recent papers, the study of PDEs has appeared to be a subject of interest, dealing with the existence and qualitative behavior of the solutions [13,14]. In this work, we consider the Partial difference Equations (PdEs) method that mimics PDEs on graphs, by replacing differential operators by difference operators on graphs.

Following these works on PdEs on graphs [12], we propose to extend the notion of mean curvature to discrete settings and to show the relation between this mean

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curvature and local and nonlocal forms of curvature in Euclidean domains. We also extend mean curvature applications to any discrete data that can be represented by graphs to solve many problems in image and manifold processing.

1.2. Short overview on level set mean curvature flow

In the last few decades, there has been increasing interest in mean curvature flows with applications in image processing (denoising, enhancement, segmentation); many papers have been devoted to its numerical algorithms. These algorithms are related to finite difference methods on uniform grids, threshold dynamics [22] and mathematical morphology using Min/Max operators on game theoretical approach, see [4–6] for more details.

The level set formulation to describe the curve evolution has been introduced by Osher-Sethian [1]. It provides well-known advantages such as treating self-intersections or topological changes and can be easily extended to $\mathbb{R}^d$ with $d \geq 1$. Given a parametrized curve $\Gamma : [0,1] \to \Omega$, evolving on a domain $\Omega \subset \mathbb{R}^d$ due to the effect of a scalar field $\mathcal{F} : \Omega \to \mathbb{R}$. The level set method aims to find a function $f(x,t)$ such that at each time t the evolving curve Γ_t can be provided by the 0-level set of $f(x,t)$. In other words $\Gamma_t = \{x | f(x,t) = 0\}$ and the curve evolution can be done solving

$$\frac{\partial f}{\partial t} = \mathcal{F} |\nabla f(x,t)|,$$

with an initial condition $f(x,0) = f_0(x)$, the initial embedding of Γ. In the context of image processing, f_0 corresponds to the given noisy image or to an implicit representation of a front (surface). When the normal velocity $\mathcal{F}$ also depends on the spatial derivative of the normal vector, we obtain the following mean curvature level set equation:

$$\frac{\partial f}{\partial t} = \mathcal{K} |\nabla f(x,t)|, \tag{1}$$

where $\mathcal{K} = \mathrm{div}(\nabla f / |\nabla f|)$, the quantity $|\nabla f(x,t)|$ is the module of gradient.

1.3. Contributions

Our main contributions are as follows. We propose to define the notion of discrete weighted perimeters using a family of discrete gradients on graphs. As in the continuous setting, we introduce the notion of nonlocal curvature as the first variation of the discrete perimeters. We show that our formulation unifies both local and nonlocal notions of the curvature.

The transcription of the level set equation on graphs by replacing curvature and gradient leads to a PdE. The new numerical scheme we propose leads a morphological approach alternating dilation and erosion processes as

$$\frac{\partial f}{\partial t} = \max(k_w(u),0) |\nabla_w^+ f(u)|_p + \min(k_w(u),0) |\nabla_w^- f(u)|_p$$

where $\mathcal{K}_w, \nabla_w^+, \nabla_w^-$ are respectively the nonlocal curvature and upwind gradient on a given weighted graph $G = (V,E,w)$.

Finally, we show that our approach can deal with different types of applications including image filtering, images on mesh filtering and 3D surface smoothing.

Remark. The term nonlocal, applied to our discrete operators, is related to the non-locality of data defined on Euclidean domains (as images). Indeed, by graph construction, these operators can mimic non-local operators defined on the continuous domain. Then, this term is used as a reference to the continuous case [21] where it means that each element can interact with every other element in the domain (and not only adjacent ones), and should not be confused with the one in non-local filtering (that uses patches).

1.4. Paper organization

The rest of this paper is organized as follows. Section 2 presents a general definition of Partial difference Equations on weighted graph. Section 3 presents our new formalism of the Mean Curvature. Section 4 presents some experiments. Finally, Section 5 concludes this paper.

2. Partial difference equations on graphs

2.1. Notations and definitions

We begin briefly by reviewing some basic definitions and operators on weighted graphs. See [2,9] for more details.

Let us consider the general situation where any discrete domain can be viewed as a weighted graph. A weighted graph $G = (V,E,w)$ consists of a finite set V of N vertices and of a finite set $E \subseteq V \times V$ of edges. Let (u,v) be the edge that connects vertices u and v. An undirected graph is weighted if it is associated with a weight function $w : V \times V \to [0,1]$. The weight function represents a similarity measure between two vertices of the graph. According to the weight function, the set of edges is defined as $E = \{(u,v) | w(u,v) > 0\}$. The degree of a vertex u is defined as $\mu(u) = \sum_{v \sim u} w(u,v)$. The neighborhood of a vertex u (i.e., the set of vertices adjacent to u) is denoted $N(u)$. Notation $v \sim u$ means that the vertex v is adjacent to u. Let $\mathcal{H}(V)$ be the Hilbert space of real valued functions on the vertices of the graph. Each function $f : V \to \mathbb{R}$ of $\mathcal{H}(V)$ assigns a real value $f(u)$ to each vertex $u \in V$. Similarly, let $\mathcal{H}(E)$ be the Hilbert space of real valued functions defined on the edges of the graph. These two spaces are endowed with the following inner products: $\langle f,h \rangle_{\mathcal{H}(V)} = \sum_{u \in V} f(u) g(u) \mu(u)$ with $f,g \in \mathcal{H}(V)$, and $\langle F,H \rangle_{\mathcal{H}(E)} = \sum_{u \in V} \sum_{v \in V} F(u,v) G(u,v) w(u,v)$ where $F,G \in \mathcal{H}(E)$.

Given a function $f : V \to \mathbb{R}$, the $\mathcal{L}_p$ norm of f is given by

$$\|f\|_p = \left(\sum_{u \in V} |f(u)|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

$$\|f\|_\infty = \max_{u \in V} (|f(u)|), \quad p = \infty.$$

2.2. Difference, divergence and discrete gradients on graphs

Let us fix a weighted graph $G=(V,E,w)$. The *difference operator* $\mathcal{G}:\mathcal{H}(V)\to\mathcal{H}(E)$ is given for all $f\in\mathcal{H}(V)$ and $(u,v)\in E$ by $(\mathcal{G}f)(u,v)=(f(v)-f(u))$.

The *directional derivative* (or edge derivative) of a function f at a vertex v along an edge $e=uv$ is defined as $\partial_v f(u)=(f(v)-f(u))$.

The *adjoint operator* of the difference operator, denoted by $\mathcal{G}^*:\mathcal{H}(E)\to\mathcal{H}(V)$, is defined by $\langle d_w f,H\rangle_{\mathcal{H}(E)}=\langle f,G_w^* H\rangle_{\mathcal{H}(V)}$, with $f\in\mathcal{H}(V)$ and $H\in\mathcal{H}(E)$. Using the definitions of the inner products in $\mathcal{H}(V)$ and $\mathcal{H}(E)$ and the definition of the difference operator, we obtain easily the expression $\mathcal{G}^*$ at a vertex u: $(\mathcal{G}^*H)(u)=\sum_{v\sim u}(w(u,v)/\mu(u))(H(v,u)-H(u,v))$.

The *divergence operator*, defined by $\text{div}_w=-\mathcal{G}^*$, measures the net outflow of a function in $\mathcal{H}(E)$ at each vertex of V.

Two *weighted directional difference operators* can be defined. The weighted directional external and internal difference operators are respectively

$$(\partial_v^+ f)(u)=(f(v)-f(u))^+ \quad \text{and}$$
$$(\partial_v^- f)(u)=(f(v)-f(u))^-, \tag{2}$$

with $(x)^+=\max(0,x)$ and $(x)^-=-\min(0,x)$.

The *weighted gradient* of a function $f\in\mathcal{H}(\mathbf{V})$ at vertex u is the vector of all edge directional derivatives:

$$(\nabla_w f)(u)=(\partial_v f(u))^T_{v\in V}. \tag{3}$$

Two *discrete formulations* of weighted morphological gradients on graphs are defined. The weighted external ∇_w^+ and the internal ∇_w^- gradient operators are respectively

$$(\nabla_w^+ f)(u)=(\partial_v^+ f(u))^T_{v\in V}, \tag{4}$$

$$(\nabla_w^- f)(u)=(\partial_v^- f(u))^T_{v\in V}. \tag{5}$$

To define a notion of regularity of a function f around a vertex u, we can consider different *norms of gradients* as follows:

$$\|(\nabla_w^\pm f)(u)\|_p=\left[\sum_{v\sim u} w(u,v)((f(v)-f(u))^\pm)^p\right]^{1/p}. \tag{6}$$

$$\|(\nabla_w^\pm f)(u)\|_\infty=\max_{v\sim u}(w(u,v)((f(v)-f(u))^\pm)), \tag{7}$$

$\nabla_w^\pm$ refers to both external and internal gradients (with respect to the sign). These gradients have the following property:

$$\|(\nabla_w f)(u)\|_p^p=\|(\nabla_w^+ f)(u)\|_p^p+\|(\nabla_w^- f)(u)\|_p^p. \tag{8}$$

Moreover, with a constant weight function and $p=\infty$, Eq. (8) recovers the usual expression of algebraic morphological external and internal gradients.

These gradients are used in [9] to adapt the well-known Eikonal equation on continuous domains defined as

$$\frac{\partial f}{\partial t}(x,t)=F(x)\|(\nabla f)(x,t)\|_p, \quad F(x)\in\mathbb{R}, \tag{9}$$

to the discrete following equation on graph:

$$\frac{\partial f}{\partial t}(u,t)=F^+(u)\|\left(\nabla_w^+ f\right)(u)\|_p-F^-(u)\|\left(\nabla_w^- f\right)(u)\|_p. \tag{10}$$

This equation summarizes the dilation and erosion processes. When $F>0$, then the external gradient is used and this equation corresponds to a dilation. When $F<0$, this equation corresponds to an erosion.

3. Mean curvature on graph

In this section, we present our new definition of mean curvature on graph by introducing the nonlocal perimeters on graph. We define the mean curvature as the first variation of these perimeters. We will show that the transcription of the mean curvature equation (1) using our definition and the morphological gradients leads to a difference equation that can be solved by a simple and iterative digital algorithm involving morphological dilatation and erosion on graph.

3.1. Nonlocal perimeters and co-area formula on graph

Let $\mathcal{A}$ be a set of connected vertices with $A\subset V$. We denote $\partial^+\mathcal{A}=\{u\in\mathcal{A}^c|\exists v\in\mathcal{A}, v\sim u\}$ as the outer vertex boundary, let $\partial^-\mathcal{A}=\{u\in\mathcal{A}|\exists v\in\mathcal{A}^c, v\sim u\}$ be the inner vertex boundary where $\mathcal{A}^c$ is the complement of $\mathcal{A}$ (Fig. 1). Let $\partial_v\mathcal{A}=\partial^+\mathcal{A}\cup\partial^-\mathcal{A}$ be the symmetric vertex

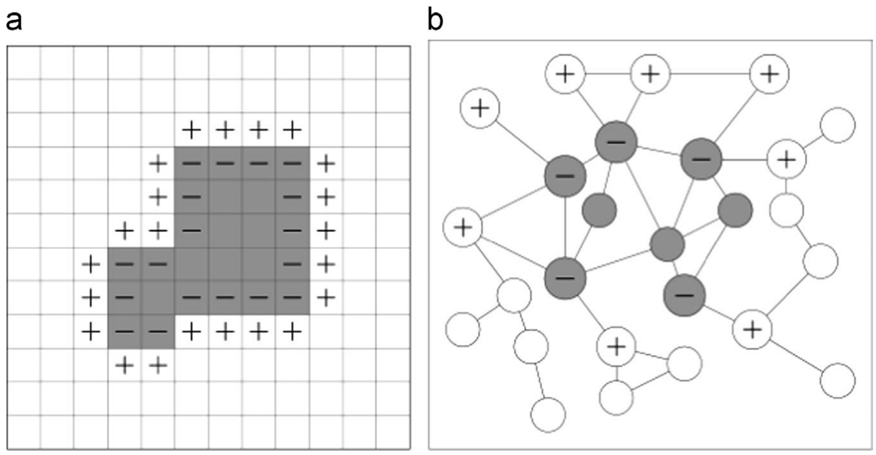

Fig. 1. Graph boundary on two different graphs. Gray vertices correspond to set $\mathcal{A}$. Plus and minus vertices are respectively outer $\partial^+\mathcal{A}$ and inner $\partial^-\mathcal{A}$ sets. (a) 4-Adjacency image grid graph. (b) Arbitrary undirected graph.

boundary. Note that $\partial^+\mathcal{A}=\partial^-(\mathcal{A}^c), \partial^-(\mathcal{A})=\partial^+(\mathcal{A}^c)$ and $\partial(\mathcal{A})= \partial(\mathcal{A}^c)$. We define the edge boundary $\partial_E\mathcal{A}=\{(u,v)\in E, u\in\mathcal{A}, v\in\mathcal{A}^c\}$. Let χ be an indicator function with

$$\chi_{\mathcal{A}}(u)=\begin{cases}1 & \text{if } u\in\mathcal{A}\\ 0 & \text{otherwise.}\end{cases}$$

The following proposition elucidates the relationship between the discrete gradient and the boundary set which are defined.

Proposition. *Let* $1\le p<\infty$ *and* $\mathcal{A}\subset V$.

$$\|\nabla_w^+\chi_{\mathcal{A}}(u)\|_p=\left(\sum_{v\in\mathcal{A}}(w(u,v))^p\right)^{1/p}\chi_{\partial^+\mathcal{A}}(u), \tag{11}$$

$$\|\nabla_w^-\chi_{\mathcal{A}}(u)\|_p=\left(\sum_{v\in\mathcal{A}^c}(w(u,v))^p\right)^{1/p}\chi_{\partial^-\mathcal{A}}(u), \tag{12}$$

$$\|\nabla_w\chi_{\mathcal{A}}(u)\|_p=\|\nabla_w^+\chi_{\mathcal{A}}(u)\|_p+\|\nabla_w^-\chi_{\mathcal{A}}(u)\|_p. \tag{13}$$

The above equations were obtained by replacing the different variables (f by $\chi_{\mathcal{A}}$) in Eqs. (6) and (8).

We see that for $u\in\mathcal{A}$, $\|\nabla_w^+\chi_{\mathcal{A}}(u)\|_p$ corresponds to the weighted number of neighbors of $u\in\mathcal{A}^c$ (equivalently the weighted numbers of the out-boundary edges between u and $\mathcal{A}^c$), while $\|\nabla_w^-\chi_{\mathcal{A}}(u)\|_p$ is the weighted number of in-boundary edges between $u\in\mathcal{A}^c$ and $\|\nabla_w\chi_{\mathcal{A}}(u)\|_p$ is the weighted number of in-boundary and out-boundary edges.

Based on the previous definitions, let us define a family of perimeters on graph.

Definition. For $0<p<\infty$ and $\mathcal{A}\subset V$, the perimeters of $\mathcal{A}$ are defined as

$$Per_{w,p}^+(\mathcal{A})=\frac{1}{2p}\sum_{u\in V}\|\nabla_w^+\chi_{\mathcal{A}}(u)\|_p, \tag{14}$$

$$Per_{w,p}^-(\mathcal{A})=\frac{1}{2p}\sum_{u\in V}\|\nabla_w^-\chi_{\mathcal{A}}(u)\|_p, \tag{15}$$

$$Per_{w,p}(\mathcal{A})=\frac{1}{p}\Big(Per_{w,p}^+(\mathcal{A})+Per_{w,p}^-(\mathcal{A})\Big). \tag{16}$$

In the case where $p=1$, it is easy to show that

$Per_{w,1}^+(\mathcal{A})=Per_{w,1}^-(\mathcal{A})$ and

$Per_{w,1}(\mathcal{A})=2Per_{w,1}^+(\mathcal{A})=\sum_{u\in V}\|\nabla_w\chi_{\mathcal{A}}\|_1$ then

$Per_{w,1}(\mathcal{A})=\sum_{u\in\mathcal{A}^c}\sum_{v\in\mathcal{A}}w(u,v)$,

which is the definition of the graph cut formulation.

As in the continuous case where the perimeter is linked to the total variation via co-area formula, we show that our proposed perimeters cover this property.

A key property of the gradient in this case is called co-area formula. This is an extension of some properties of the total variation on graph.

Proposition. *For any function* $f:V\to R$

$$Df=\int_{-\infty}^{+\infty}D\chi_{\{f>t\}}\,dt, \tag{17}$$

where D denotes any one of $|\nabla_w|_1, |\nabla_w|_1^{\pm}, |\nabla_w|_\infty^{\pm}$.

D are the functions of the form $\sum_{u\in V}w(u,v)(f(v)-f(u))^{\pm}$ *and* $\sum_{u\in V}(w(u,v)|f(v)-f(u)|$, *where the weights* $w(u,v)$ *are non-negative.*

Proof. This proposition just follows easily from $|a-b|=\int_{-\infty}^{+\infty}|\chi_{\{a>t\}}-\chi_{\{b>t\}}|\,dt$, and $(a-b)^{\pm}=\int_{-\infty}^{+\infty}(\chi_{\{a>t\}}-\chi_{\{b>t\}})^{\pm}\,dt$, where a and $b\in R$.

The above proposition allows to recover the following definitions which can be used to relax many problems of optimization involving the discrete perimeters:

$$J_{w,1}^+(f)=\int_{-\infty}^{+\infty}J_{w,1}^+(\chi_{\{f(u)>t\}})\,dt = \int_{-\infty}^{+\infty}Per_{w,1}^+(\chi_{\{f(u)>t\}})\,dt. \tag{18}$$

$$J_{w,1}^-(f)=\int_{-\infty}^{+\infty}J_{w,1}^-(\chi_{\{f(u)>t\}})\,dt = \int_{-\infty}^{+\infty}Per_{w,1}^-(\chi_{\{f(u)>t\}})\,dt. \tag{19}$$

$$J_{w,1}(f)=\int_{-\infty}^{+\infty}J_{w,1}(\chi_{\{f(u)>t\}})\,dt = \int_{-\infty}^{+\infty}Per_{w,1}(\chi_{\{f(u)>t\}})\,dt, \tag{20}$$

where $J_{w,1}$ is a regularization functional that extends the total variation on graph and is defined as

$$J_{w,1}^+f(u)=\frac{1}{2}\sum_{u\in V}\|(\nabla_w^+f)(u)\|_1, \tag{21}$$

$$J_{w,1}^-f(u)=\frac{1}{2}\sum_{u\in V}\|(\nabla_w^-f)(u)\|_1, \tag{22}$$

$$J_{w,1}f(u)=\sum_{u\in V}\|(\nabla_w f)(u)\|_1. \tag{23}$$

As in the continuous domains, we define the mean curvature as the first variation of the perimeters that we have just defined (16). □

Definition. Let $u_0\in\partial\mathcal{A}=\partial^+\mathcal{A}\cup\partial^-\mathcal{A}$.

For $u_0\in\partial^+\mathcal{A}$, the mean curvature of u_0 is defined as

$$\mathcal{K}_w^+(u_0,\mathcal{A})=\frac{Per_{w,1}^+(\mathcal{A}\cup\{u_0\})-Per_{w,1}^+(\mathcal{A})}{\mu(u_0)}, \tag{24}$$

where $\mu(u_0)$ define the degree of a vertex u_0. And for $u_0\in\partial^-\mathcal{A}$, the mean curvature of u_0 is defined as

$$\mathcal{K}_w^-(u_0,\mathcal{A})=\frac{Per_{w,1}^-(\mathcal{A})-Per_{w,1}^-(\mathcal{A}-\{u_0\})}{\mu(u_0)}. \tag{25}$$

By replacing the variables ($\mathcal{A}$ by $\mathcal{A}\cup\{u_0\}$) in Eq. (14), it can be rewritten as

$$Per_{w,1}^+(\mathcal{A}\cup\{u_0\})=\sum_{v\in((\mathcal{A}\cup\{u_0\})^c}\sum_{v\in((\mathcal{A}\cup\{u_0\})}\gamma(u,v) = \sum_{v\in(\mathcal{A}^c-\{u_0\})}\sum_{v\in(\mathcal{A}\cup\{u_0\})}\gamma(u,v). \tag{26}$$

It is easy to show that

$$Per^+_{w,1}(\mathcal{A} \cup \{u_0\}) - Per^+_{w,1}(\mathcal{A}) = \sum_{v \in \mathcal{A}^c} \gamma(u_0, v) - \sum_{v \in \mathcal{A}} \gamma(u_0, v), \tag{27}$$

then the mean curvature $\mathcal{K}^+_w$ of u_0 is rewritten as

$$\mathcal{K}^+_w(u_0, \mathcal{A}) = \frac{\sum_{v \in \mathcal{A}^c} \gamma(u_0, v) - \sum_{v \in \mathcal{A}} \gamma(u_0, v)}{\mu(u_0)}. \tag{28}$$

Similarly, the mean curvature $\mathcal{K}^-_w$ of u_0 is rewritten as

$$\mathcal{K}^-_w(u_0, \mathcal{A}) = \frac{\sum_{v \in \mathcal{A}^c} \gamma(u_0, v) - \sum_{v \in \mathcal{A}} \gamma(u_0, v)}{\mu(u_0)}. \tag{29}$$

Then, for $u_0 \in \partial\mathcal{A}$, the mean curvature is defined as

$$\mathcal{K}_{w,1}(u_0, \mathcal{A}) = \frac{\sum_{v \in \mathcal{A}} w(u_0, v) - \sum_{v \in \mathcal{A}^c} w(u_0, v)}{\mu(u_0)}. \tag{30}$$

Based on this definition, we can extend the notion of curvature to any function f on a graph by considering its level sets.

Let $f: V \to \mathbb{R}$ and $u \in V$. The mean curvature $\mathcal{K}_{w,1}$ of f at u on a graph is defined as

$$\mathcal{K}_{w,1}(u, f) = \mathcal{K}_{w,1}(u, \{f(v) \geq f(u)\}) \tag{31}$$

$$\mathcal{K}_{w,1}(u, f) = \frac{\sum_{f(v)-f(u) \geq 0} w(u, v) - \sum_{f(v)-f(u) < 0} w(u, v)}{\mu(u)} \tag{32}$$

$$\mathcal{K}_{w,1}(u, f) = \frac{\sum_{u \in V} w(u, v)\, \mathrm{sign}(f(v) - f(u))}{\mu(u)}, \tag{33}$$

with

$$\mathrm{sign}(r) = \begin{cases} 1 & \text{if } r \geq 0 \\ -1 & \text{otherwise.} \end{cases}$$

3.2. Connection with nonlocal mean curvature and Euclidean domains

In this section, we show that our definition of mean curvature is linked to the notion of fractional mean curvature on Euclidean domain introduced in [10]. The notion of the fractional perimeter (s-perimeter) and the corresponding minimization problem was introduced in [11]. The s-perimeter of $A \subset \mathbb{R}^n$ is defined as

$$Per_s(\mathcal{A}) = c_n \int_{\mathcal{A}} \int_{\mathcal{A}^c} \frac{1}{|x-y|^{n+s}}\, dx\, dy, \tag{34}$$

where x and y $\in \mathcal{A}$ and c_n is a normalization constant.

The main idea of the s-perimeter is that any point inside $\mathcal{A}$ interacts with any point outside $\mathcal{A}$. The continuous fractional curvature is defined formally as the first variation of these s-perimeters as follows:

$$\mathcal{K}(x, \mathcal{A}) = c_n \int_{\mathcal{A}} \frac{\chi_{\mathcal{A}}(y) - \chi_{\mathcal{A}^c}(y)}{|x-y|^{n+s}}\, dy = c_n \int_{\mathcal{A}} \frac{1}{|x-y|^{n+s}}\, dy - c_n \int_{\mathcal{A}^c} \frac{1}{|x-y|^{n+s}}\, dy. \tag{35}$$

The above equation can be interpreted as a continuous version of our proposed definition. Let us consider a nonlocal Euclidean graph $G = (V, E, w)$ with $V = \mathbb{R}^n$, $\mathcal{A} \subset V$, $E = \{(x, y) \in V \times V / w(x, y) > 0\}$.

For

$$w(x, y) = \begin{cases} \dfrac{1}{|x-y|^{n+s}} & \text{with } 0 < s < 1 \\ 0 & \text{otherwise,} \end{cases}$$

our mean curvature is defined as

$$\mathcal{K}_{w,1}(x, \mathcal{A}) = \frac{\sum_{y \in \mathcal{A}} w(x, y) - \sum_{y \in \mathcal{A}^c} w(x, y)}{\mu(x)}. \tag{36}$$

One can see that the above equation corresponds to the continuous version of Eq. (35) if we consider $\mu(x) = c_n$.

Now, we consider a local Euclidean graph G with a weight function defined as follows:

$$w(x, y) = \begin{cases} 1 & \text{with } y \in B_\xi(x) \\ 0 & \text{otherwise,} \end{cases}$$

where $B_\xi(x)$ is a ball centered on x and with radius ξ. Then, the mean curvature can be rewritten as

$$\mathcal{K}_{w,1}(x, \mathcal{A}) = \frac{\sum_{y \in \mathcal{A} \cap B_\xi(x)} 1 - \sum_{y \in \mathcal{A}^c \cap B_\xi(x)} 1}{\mu(x)} = \frac{|\mathcal{A} \cap B_\xi(x)| - |\mathcal{A}^c \cap B_\xi(x)|}{|B_\xi(x)|}, \tag{37}$$

where $|X|$ is the cardinal of X. This equation is an approximation of the continuous local curvature.

3.3. Morphological scheme for mean curvature flows

In this section, we present a general numerical scheme for the mean curvature flows on graphs. This scheme can deal with graphs of different topologies and with different norms. We also provide an explicit scheme for the case of L_∞ norm as well as a morphological interpretation.

Based on the definition of our mean curvature and the transcription of Eq. (1), our formulation can be expressed as follows:

$$\begin{cases} \dfrac{\partial f}{\partial t}(u) = \mathcal{K}^+_w(f(u)) \| (\nabla^+_w f)(u) \|_p - \mathcal{K}^-_w(f(u)) \| (\nabla^-_w f)(u) \|_p \\ f(u, 0) = f_0(u), \end{cases} \tag{38}$$

where $\mathcal{K}^+_w(x) = (\mathcal{K}_w(x))^+$ and $\mathcal{K}^-_w(x) = (\mathcal{K}_w(x))^-$.

Now, let us show that this iterative process corresponds to an alternate dilation and erosion type filter depending on the sign of $\mathcal{K}_w$. In particular, when $\mathcal{K}_w > 0$, Eq. (38) becomes

$$\begin{cases} \dfrac{\partial f}{\partial t}(u) = \mathcal{K}^+_w(f(u)) \| (\nabla^+_w f)(u) \|_p \\ f(u, 0) = f_0(u), \end{cases} \tag{39}$$

that corresponds to a discrete dilation process.

Similarly, when $\mathcal{K}_w < 0$, Eq. (38) becomes

$$\begin{cases} \dfrac{\partial f}{\partial t}(u) = \mathcal{K}^-_w(f(u)) \| (\nabla^-_w f)(u) \|_p \\ f(u, 0) = f_0(u), \end{cases} \tag{40}$$

that corresponds to a discrete erosion process.

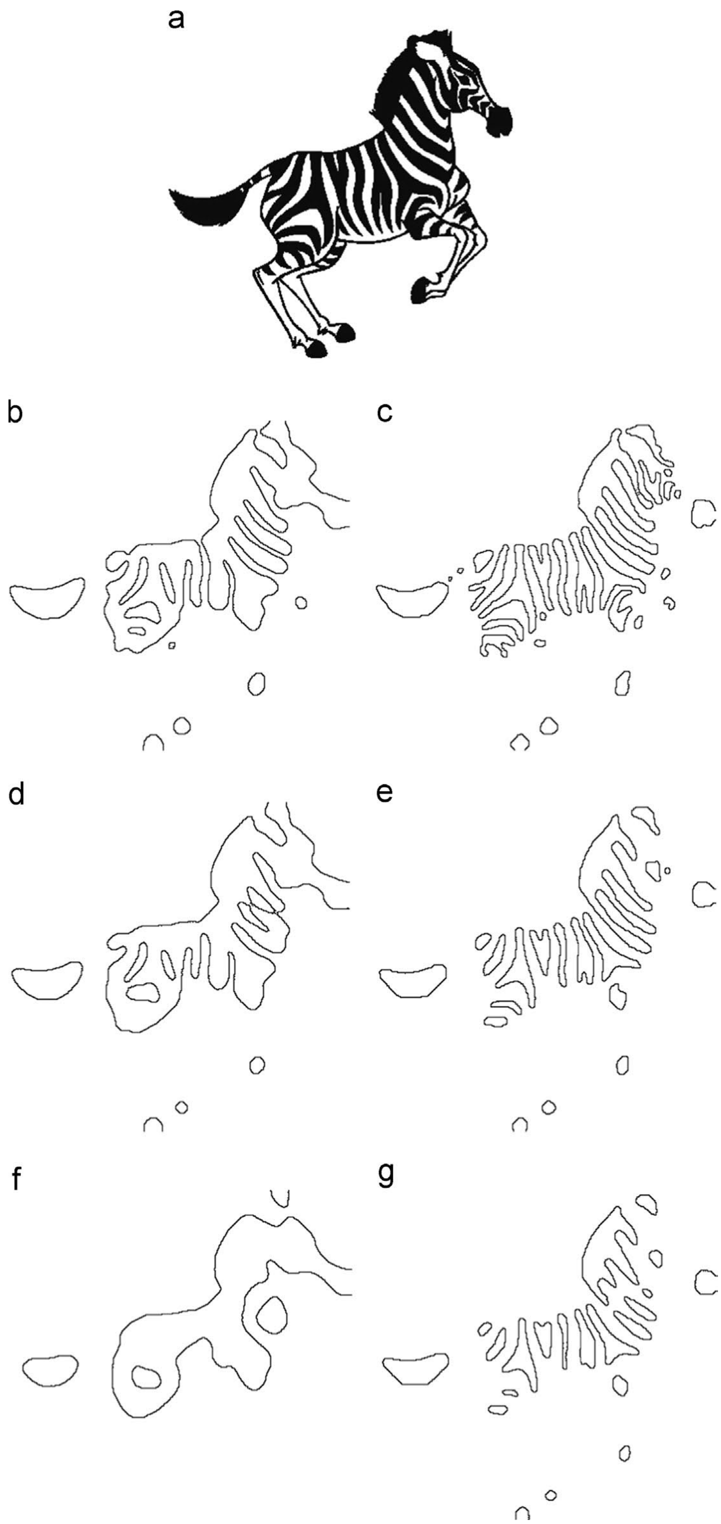

Fig. 2. Mean curvature evolution of a zebra curve. (a) presents the initial zebra image. (b, d, f) present the curve motion by mean curvature on a local structure graph after 5, 10 and 20 iterations respectively. (c, e, g) present the curve motion by mean curvature on a nonlocal graph structure after 5, 10 and 20 iterations respectively.

In the case where $p=\infty$, Eq. (38) becomes

$$\begin{cases} \frac{\partial f}{\partial t}(u)=\mathcal{K}_w^+(f(u))\|(\nabla_w^+ f)(u)\|_\infty - \mathcal{K}_w^-(f(u))\|(\nabla_w^- f)(u)\|_\infty \\ f(u,0)=f_0(u). \end{cases} \quad (41)$$

To derive morphological scheme for the solution of Eq. (41), we introduce two operators, nonlocal dilation (NLD) and nonlocal erosion (NLE) that are defined respectively as

$$NLD(f(u))=f(u)+\|(\nabla_w^+ f)(u)\|_\infty. \quad (42)$$

$$NLE(f(u))=f(u)-\|(\nabla_w^- f)(u)\|_\infty. \quad (43)$$

The time variable can be discretized using explicit Euler method as

$$\frac{\partial f}{\partial t}(u)=\frac{f^{n+1}(u)-f^n(u)}{\Delta t}, \quad (44)$$

where $f^n(u)=f(u,n\Delta t)$ and Eq. (41) can be rewritten as the following iterative equation:

$$f^{n+1}(u)-f^n(u)=\Delta t(\mathcal{K}_w^+ f^n(u)\|(\nabla_w^+ f^n)(u)\|_\infty) - \Delta t(\mathcal{K}_w^- f^n(u)\|(\nabla_w^- f^n)(u)\|_\infty) \quad (45)$$

The above equation can be rewritten using the *NLD* and *NLE* definitions as

$$\begin{aligned} f^{n+1}(u) &= \Delta t(\mathcal{K}_w^+ f^n(u) NLD(f^n(u)) - \Delta t(\mathcal{K}_w^+ f^n(u)) f^n(u) \\ &\quad - \Delta t(\mathcal{K}_w^- f^n(u)) f^n(u) \\ &\quad + \Delta t(\mathcal{K}_w^- f^n(u)) NLE(f^n(u)) + f^n(u) \\ &= f^n(u)(1-\Delta t[\mathcal{K}_w^+(f^n(u))+\mathcal{K}_w^-(f^n(u))]) \\ &\quad + \Delta t \mathcal{K}_w^+(f^n(u)) NLD(f^n(u)) \\ &\quad + \Delta t \mathcal{K}_w^-(f^n(u)) NLE(f^n(u)). \end{aligned} \quad (46)$$

One can remark that contrary to the PDEs case, no spatial discretization is needed thanks to derivatives directly expressed in a discrete form.

In the case where $\mathcal{K}_w > 0$ and $\Delta t < 1/\mathcal{K}_w$, Eq. (46) summarizes the averaging between the initial function and the NLD operator and can be rewritten as

$$\begin{aligned} f^{n+1}(u) &= f^n(u)(1-\Delta t \mathcal{K}_w^+(f^n(u))) \\ &\quad + \Delta t \mathcal{K}_w^+(f^n(u)) NLD(f^n(u)). \end{aligned} \quad (47)$$

In the case where $1-\Delta t \mathcal{K}_w^+ = 0$, Eq. (47) can be interpreted as an iterative nonlocal dilation (NLD) process.

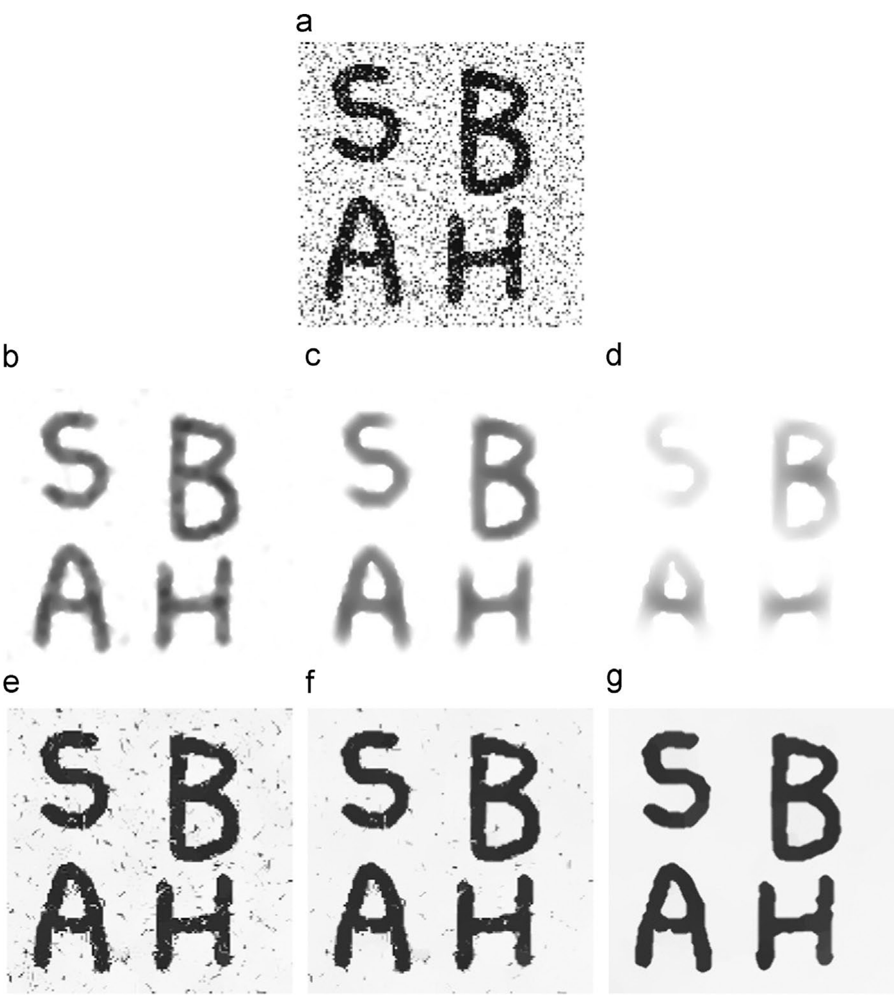

Fig. 3. Noisy image filtering. (a) presents the initial image, (b, c, d) present the filtering using mean curvature on a local graph structure after 5, 10 and 20 iterations respectively. (e, f, g) present the filtering using mean curvature on a nonlocal graph structure after 5, 10, 20 iterations respectively.

Fig. 4. Noisy image filtering. (a) presents the initial image, (b) presents a noisy Lena with sigma=20, (c) presents a noisy Lena with sigma=25, (d, e) present the filtered Lena using our mean curvature approach. (f) and (g) present respectively a colored noisy Lena image and the filtering result using our mean curvature. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

Similarly, in the case where $\mathcal{K}_w < 0$ and $\Delta t < 1/\mathcal{K}_w$, Eq. (46) summarizes the averaging between the initial function and NLE operator, and can be interpreted as an iterative NLE process.

In the case where $\mathcal{K}_w \neq 0$ and $1 - \Delta t|\mathcal{K}_w| \geq 0$, Eq. (47) summarizes an average filtering process interpolation between an image and either a nonlocal dilation or a nonlocal erosion process.

At each step of this algorithm, the new value at vertex u depends only on its value at step n and the existing values in its neighborhood.

In the particular case where $w=1$ (unweighted graph) and $u \sim u$ and $\Delta t = 1/|\mathcal{K}_1|$, Eq. (46) can be rewritten as the following iterative equation:

$$f^{n+1}(n) = \begin{cases} \max\limits_{u \sim u} f^n(u) & \text{if } \mathcal{K}_1 > 0 \\ \min\limits_{u \sim u} f^n(u) & \text{if } \mathcal{K}_1 < 0 \\ f^n(u) & \text{otherwise} \end{cases}$$

that corresponds to alternate dilation and erosion processes and that represents a new class of shock filter [15].

Fig. 5. Noisy image filtering. (a) presents the initial image, (b, c, d) present the filtering using mean curvature on a local graph structure after 5, 10 and 20 iterations respectively. (e, f) present the filtering using mean curvature on a nonlocal graph structure after 5, 10 iterations respectively, (g) presents the final filtered image with psnr=33.4928.

Particular case: In the case where we have an unweighted graph $G=(V,E,w)$. Our formulation recovers the classical algebraic flat morphological dilation formulation over graphs. The *NLD* is rewritten as

$$f^{n+1}(u)=f^n(u)+\max_{v\sim u}(\max(0,(f^n(v)-f^n(u)))). \tag{48}$$

If $f^n(v)-f^n(u)\le 0$ then $f^{n+1}=f^n$. If $f^n(v)-f^n(u)>0$ then we obtain $f^{n+1}(u)=f^n(u)+\max_{v\sim u}(0,(f^n(v)-f^n(u))=f^n(u)+\max_{v\sim u}(f^n(v))-f^n(u)$. For both cases, by considering that the neighborhood of vertex u includes u itself, we recover the classical algebraic dilation over graphs:

$$f^{n+1}(u)=\max_{v\sim u}(f^n(v)). \tag{49}$$

In this case, the structuring element is provided by the graph topology and the vertices' neighborhoods. Similarly, we can show that the NLE corresponds to the following classical algebraic erosion over graphs:

$$f^{n+1}(u)=\min_{v\sim u}(f^n(v)). \tag{50}$$

In the case where we have an unweighted grid graph example $(w=1)$, the NLD and NLE correspond to the

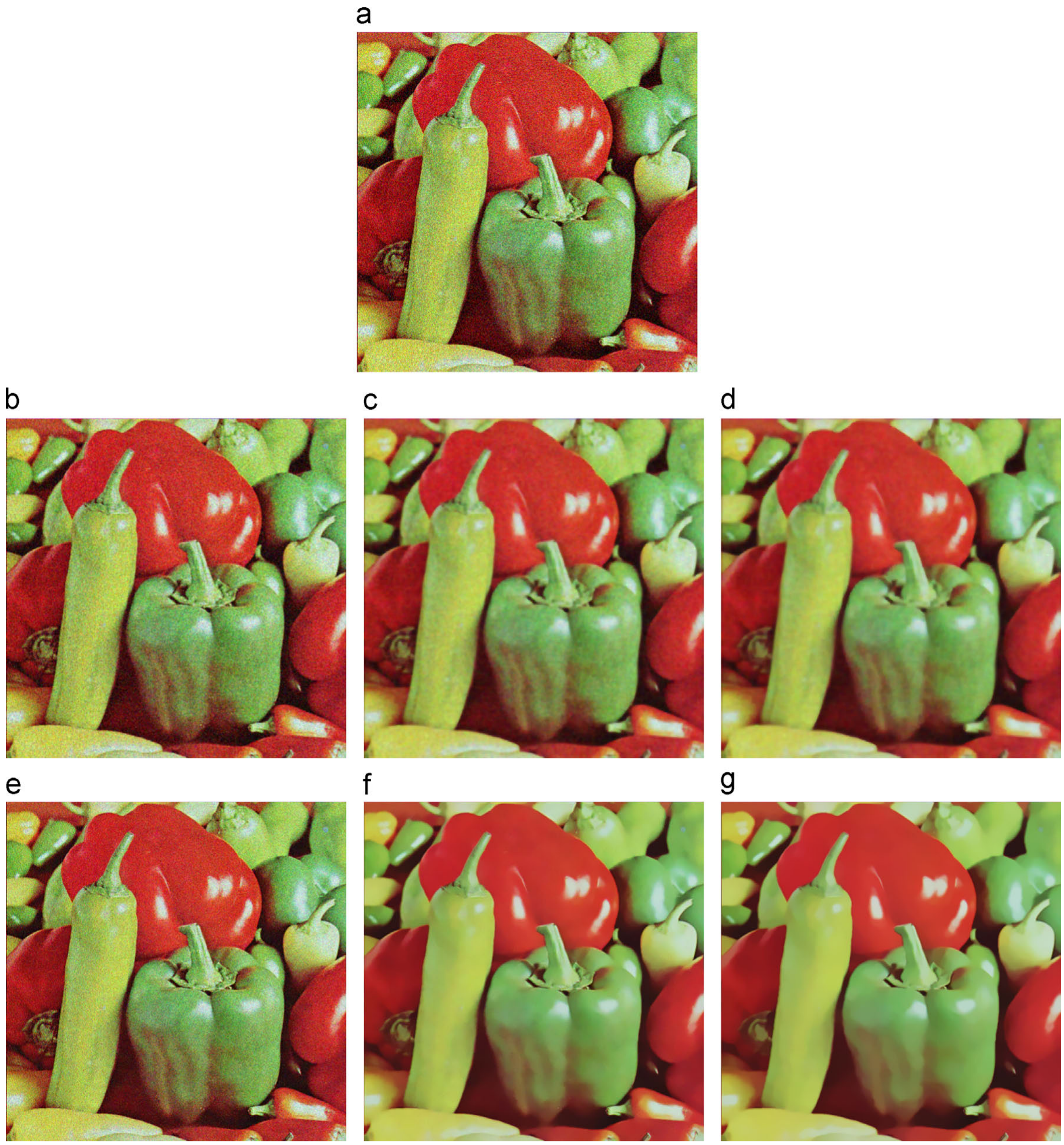

Fig. 6. Noisy image filtering. (a) presents the initial image, (b, c, d) present the filtering using mean curvature on a local graph structure after 5, 10 and 20 iterations respectively. (e, f, g) present the filtering using mean curvature on a nonlocal graph structure after 5, 10, and 20 iterations respectively.

classical dilation and erosion. Otherwise, for a weighted grid graph where the weight function depends on the image details, our scheme can be seen as a conditional adaptive dilation and erosion depends on the sign of $\mathcal{K}_w$.

3.4. Some properties

Our iterative scheme can be written as

$$f^{n+1}(u) = f^n(u)(1-\Delta t(|\mathcal{K}_w|(f^n(u)))) + \Delta t \mathcal{K}_w^+(f^n(u)) NLD(f^n(u)) + \Delta t \mathcal{K}_w^-(f^n(u)) NLE(f^n(u)). \quad (51)$$

In the case where $\Delta t = 1/|\mathcal{K}_w|$, our iterative scheme can be rewritten as

$$f^{n+1}(u) = \frac{\mathcal{K}_w^+(f^n(u))}{|\mathcal{K}_w|} NLD(f^n(u)) + \frac{\mathcal{K}_w^-(f^n(u))}{|\mathcal{K}_w|}(f^n(u)) NLE(f^n(u)) \quad (52)$$

The above equation can be rewritten as a nonlocal average operator as

$$f^{n+1}(u) = \text{sign}(\mathcal{K}_w^+(f^n(u)) NLD(f^n(u))) + \text{sign}(\mathcal{K}_w^-(f^n(u)) NLE(f^n(u))) = NLA(f^n)(u). \quad (53)$$

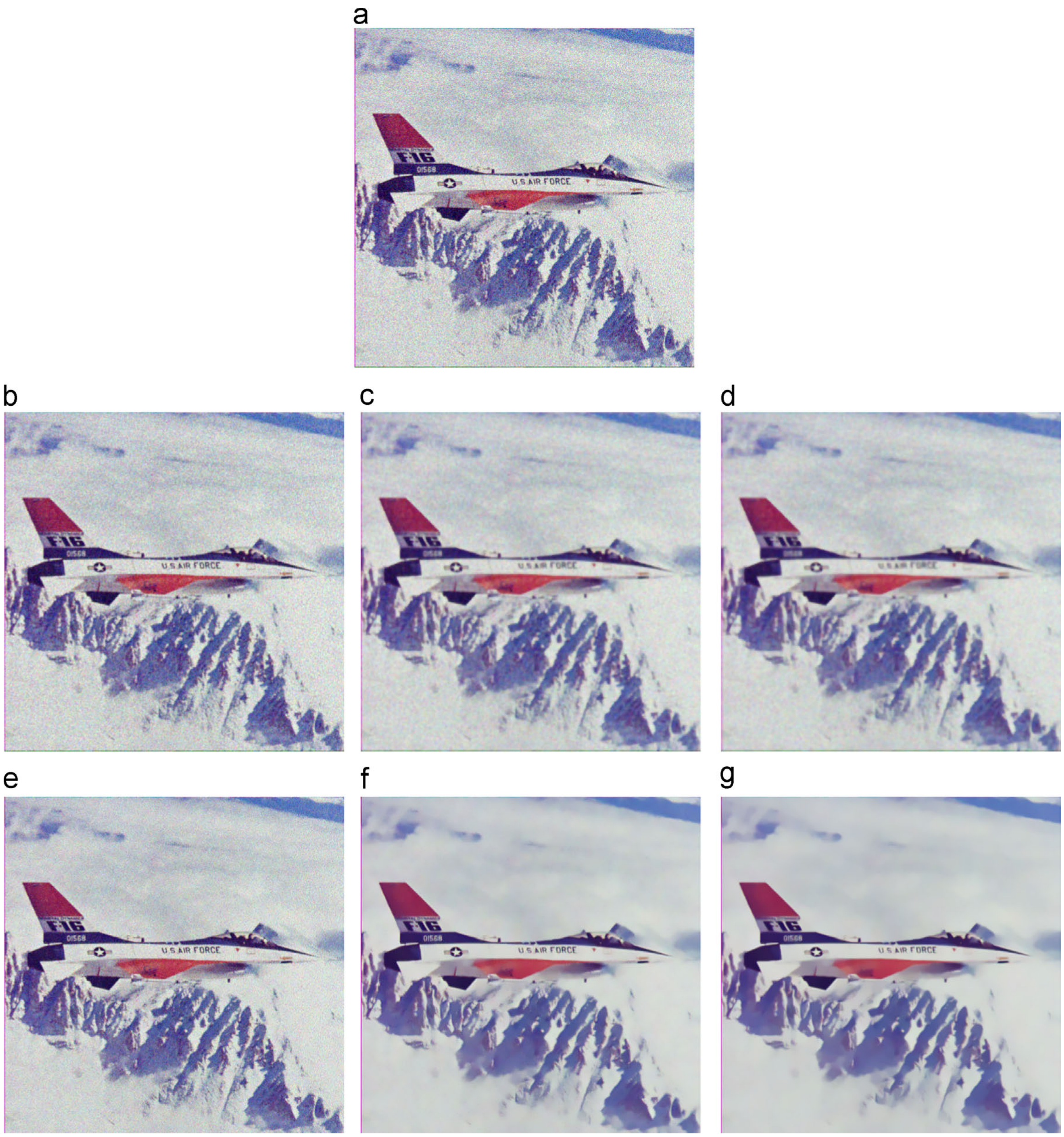

Fig. 7. Noisy image filtering. (a) presents the initial image, (b, c, d) present the filtering using mean curvature on a local graph structure after 5, 10 and 20 iterations respectively. (e, f, g) present the filtering using mean curvature on a nonlocal graph structure after 5, 10, and 20 iterations respectively.

We first show that this scheme satisfies the Minimum–Maximum Principle (MMP).

Proposition. *Our scheme (52) satisfies the MMP.*

Proof. Let $m = \min_{u \in V}(f(u))$ and $M = \max_{u \in V}(f(u))$. By definitions, we know that our nonlocal operators NLE, NLD respectively satisfy

$$m \le NLE(f^0(u)) \le M, \quad \forall u \in V,$$
$$m \le NLD(f^0(u)) \le M, \quad \forall u \in V. \tag{54}$$

According to these inequalities, we can then write

$$m \le \text{sign}(\mathcal{K}_w^+(f^0(u))NLD(f^0(u))) + \text{sign}(\mathcal{K}_w^-(f^0(u))NLE(f^0(u)))$$
$$\le M, \quad \forall u \in V, \tag{55}$$

and

$$m \le NLA(f^0)(u) \le M, \quad \forall u \in V. \tag{56}$$

Finally, by this introduction, this relation can be generalized to any time step n.

The scheme is thus stable and corresponds to a nonlocal filtering process that alternates dilation and erosion. Considering a graph $G = (V, E, w)$ and a function $f_0 \in \mathcal{H}(V)$, a simple filtering process can then be written using the following algorithm:

1. Vertices are ordered linearly. We have $u_1 < u_2 < \cdots < u_n$.
2. The algorithm is initialized with $f^0 = f_0$.
3. For every $k=0,\ldots,N$ do $f^{n+1}(u_k) = NLA(f^n)(u_k)$. □

Proposition. *If the above filtering process converges to a function f^*, then f^* satisfies $\mathcal{K}_w^+(f^*(u))\|(\nabla_w^+ f^*)(u)\|_\infty - \mathcal{K}_w^-(f^*(u))\|(\nabla_w^- f^*)(u)\|_\infty = 0$, $\forall u \in V$.*

Proof. Let f^* be the limit of scheme (52), then we have

$$f^*(u) = NLA(f^*)(u)$$
$$f^*(u) = f^*(u) + \text{sign}(\mathcal{K}_w^+(f^*(u))NLD(f^*(u)))$$
$$+ \text{sign}(\mathcal{K}_w^-(f^*(u))NLE(f^*(u)))$$
$$f^*(u) = f^*(u) + \mathcal{K}_w^+(f^*(u))\|(\nabla_w^+ f^*)(u)\|_\infty$$
$$- \mathcal{K}_w^-(f^*(u))\|(\nabla_w^- f^*)(u)\|_\infty$$
$$0 = \mathcal{K}_w^+(f^*(u))\|(\nabla_w^+ f^*)(u)\|_\infty - \mathcal{K}_w^-(f^*(u))\|(\nabla_w^- f^*)(u)\|_\infty. \quad \square \tag{57}$$

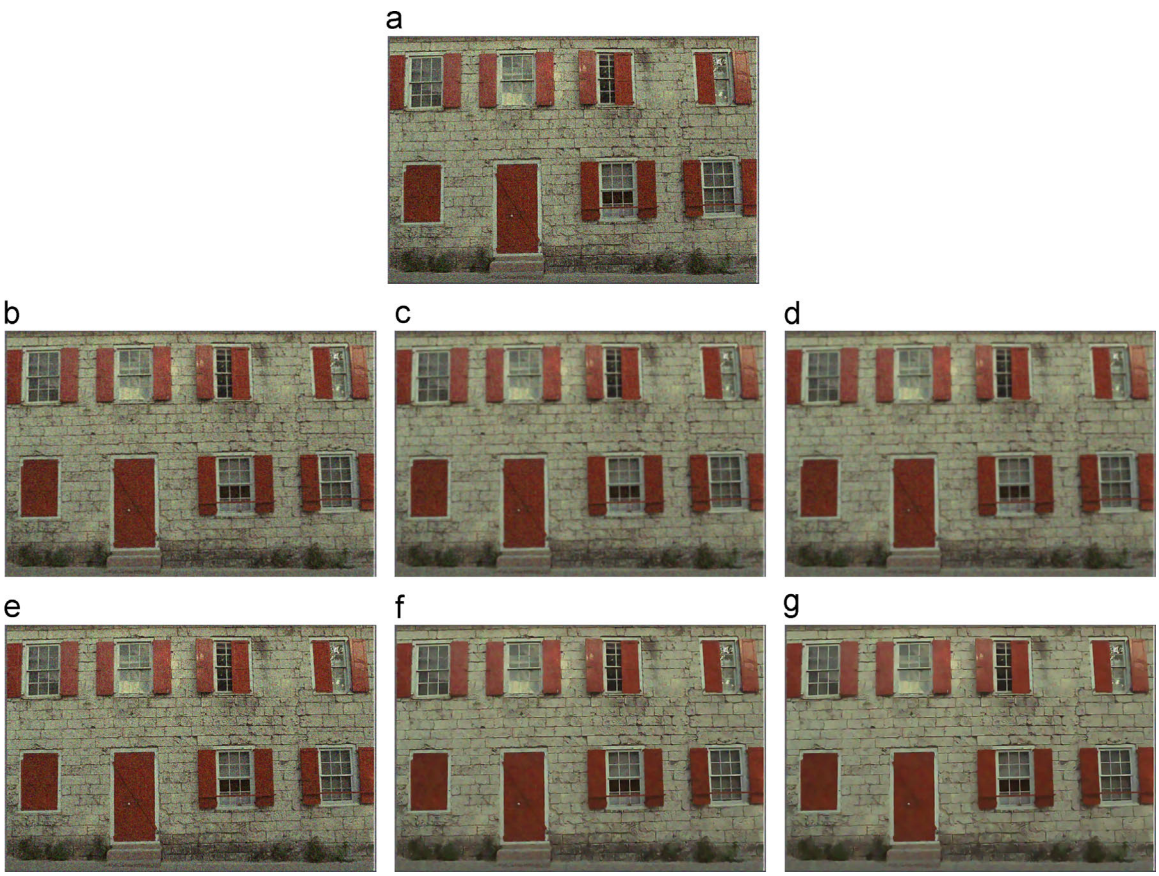

Fig. 8. Noisy image filtering. (a) presents the initial image, (b, c, d) present the filtering using mean curvature on a local graph structure after 5, 10 and 20 iterations respectively. (e, f, g) present the filtering using mean curvature on a nonlocal graph structure after 5, 10, and 20 iterations respectively.

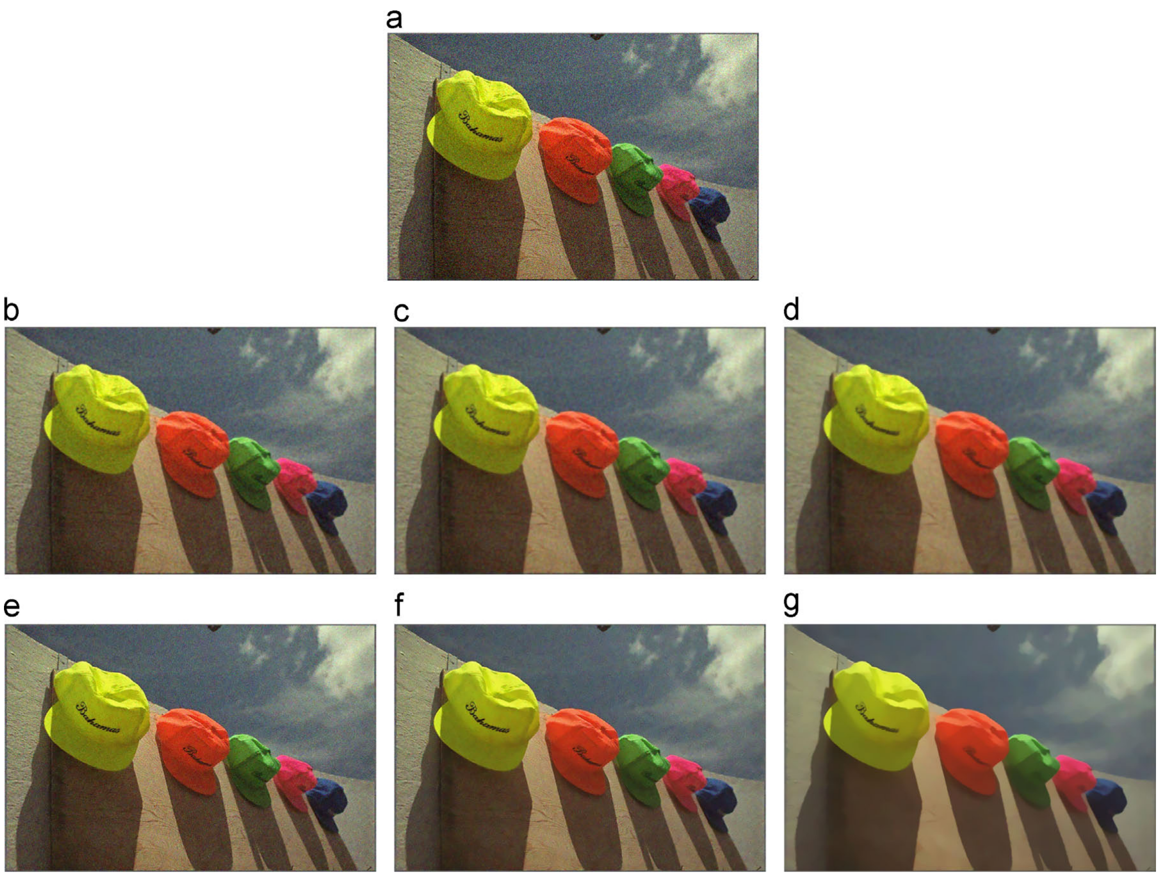

Fig. 9. Noisy image filtering. (a) presents the initial image, (b, c, d) present the filtering using mean curvature on a local graph structure after 5, 10 and 20 iterations respectively. (e, f, g) present the filtering using mean curvature on a nonlocal graph structure after 5, 10, and 20 iterations respectively.

Table 1
Filtering methods of noisy Lena image psnr comparison with different values of sigma.

sigma	**BM3D**	**NLm**	**LPA-ICI**	**BLSGSM**	**SA-DCT**	**Our method**
20	33.05	31.51	30.7	32.69	32.63	32.9158
25	32.08	30.36	29.66	31.71	31.66	31.7535

4. Applications

The proposed formulation of the mean curvature flow equation can be used to process any function defined on vertices of a graph or on any arbitrary discrete domain.

This section illustrates the potentialities of our formulation through examples of image filtering, image on surface filtering and 3D surface smoothing. Different graph structures and weight functions are also used to show the flexibility of our approach. The objective of the following experiments is not to solve a particular application. They only illustrate the potential and the behavior of our mean curvature definition formulation.

4.1. Weighted graph construction

Any discrete domain can be represented by a weighted graph where functions of $\mathcal{H}(V)$ represent the data to process. In the general case, an unorganized set of points $V \subset \mathbb{R}^n$ can be seen as a function $f^0: V \subset R^n \to R^n$. Then, constructing a graph from this data consists in defining the set of edges E by modeling the neighborhood. It is based on a similarity relationship between and a pairwise distance measure $d: V \times V \to R^+$.

In this paper, we focus on two particular graphs: the grid graphs and the k-nearest neighbor graphs. Grid graphs are the natural structure to describe an image with a graph. Each pixel is connected by an edge to its adjacent pixels. The k nearest neighbors graph, noted k-NNG, is a weighted graph where each vertex $u \in V$ is connected to its k nearest neighbors which have the smallest distance measure toward u according to function d.

Then, the weight function w can be defined using usual similarity functions depending on application and graph topology, and satisfies

$$w(u,v) = \begin{cases} g(u,v) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

where g is one of the following similarity functions:

$$g_0(u,v)=1,$$
$$g_1(u,v)=e^{(-d(f^0(u),f^0(v))/\sigma^2)} \quad \text{with } \sigma>0. \tag{58}$$

4.2. Mean curvature for shape evolution

Let f_0 be a level set function that represents our initial data where $f_0=\chi_{\Omega_0}-\chi_{\Omega_0^c}$, with $\chi: V\to\{0,1\}$ as the indicator function and Ω_0^c as the complement of Ω_0. In other words f equals 1 in Ω_0 and -1 in its complement. First in Fig. 2, we show the evolution of the zebra curve under the effect of mean curvature flow using local and nonlocal graph structures. The first column presents the evolution of the zebra curve using a local graph (i.e., 4-adjacency grid graph) with a constant weight function $w(u,v)=1$ at different steps. The second column presents the same results but on a nonlocal graph. In this example, the graph is constructed as a 16-adjacency grid graph with the

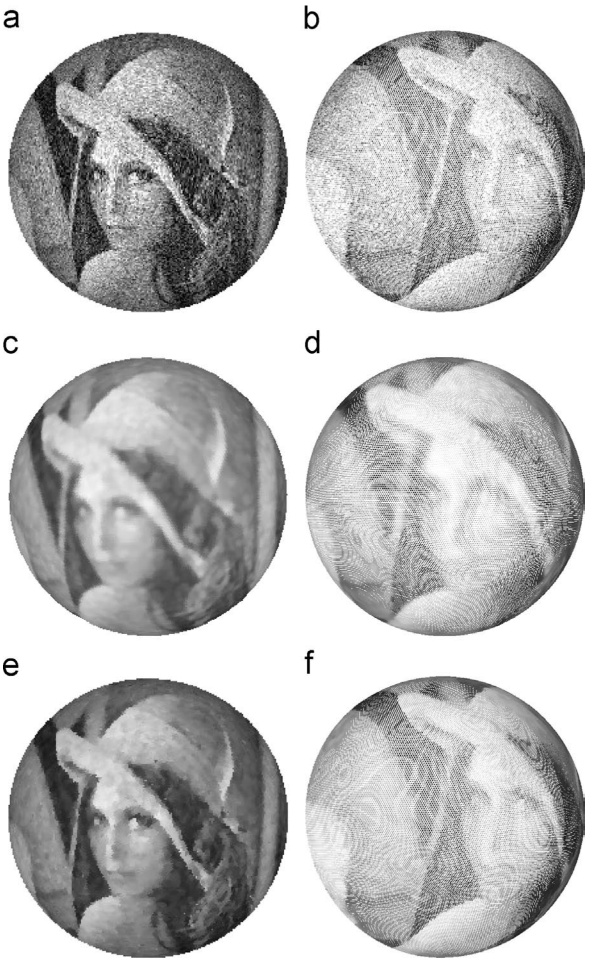

Fig. 10. Image on mesh filtering. (a) presents the initial noisy image on surface, (b) presents a zoom on the point clouds of the initial model, (c) presents the filtered model by mean curvature on local graph after 30 iterations with $w=1$, (d) presents a zoom on the point clouds of the filtered model, (e) presents the filtered model by mean curvature after 30 iterations with $\sigma=20$, and (f) presents a zoom on the point clouds of the filtered model.

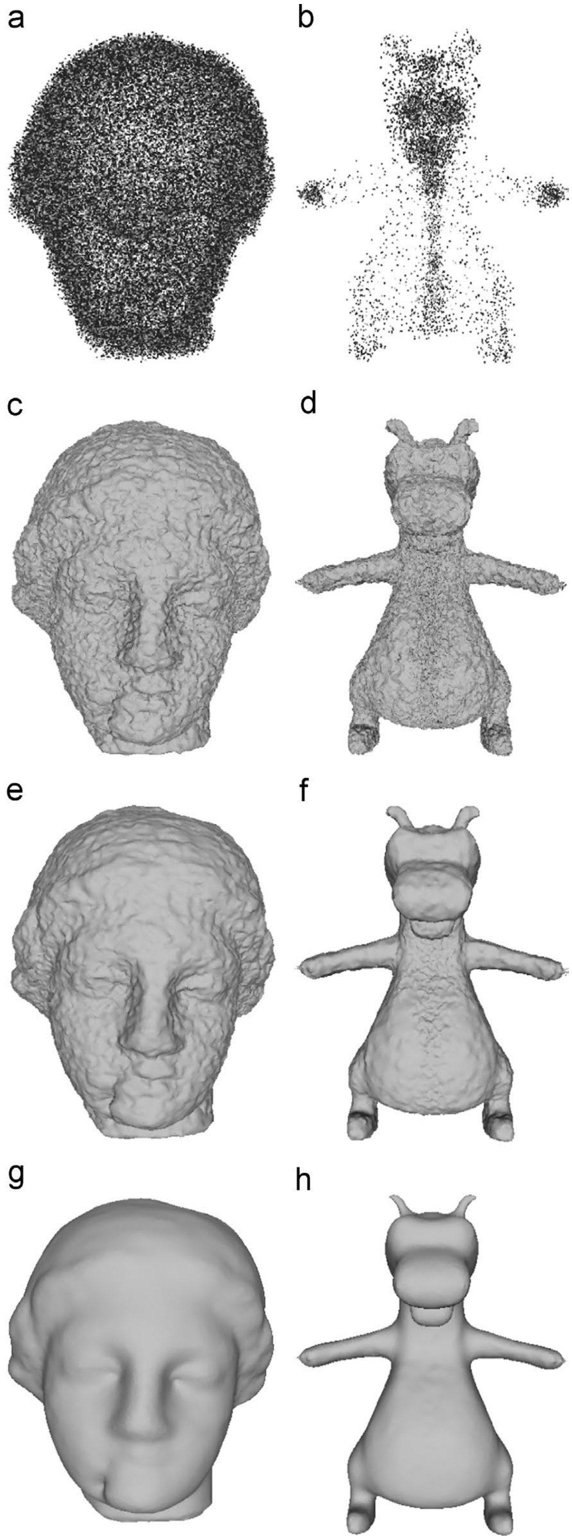

Fig. 11. 3D surface smoothing. (a, c, e, g) present respectively the initial 3D noisy Venus point clouds, the reconstructed surface, the Venus surface after 5 iterations using our mean curvature on nonlocal graph structure, and the Venus surface after 15 iterations. (b, d, f, h) present respectively the initial 3D noisy Dragon point clouds, the reconstructed surface, and the Dragon surface after 5 iterations using our mean curvature on nonlocal graph structure, the Dragon surface after 15 iterations.

weight function $w(u,v)=e^{(\|u-v\|/\sigma^2)}$ and $\sigma=10$. One can see that the motion of the curve with a nonlocal graph better preserves the global form of the initial zebra curve.

4.3. Mean curvature for image filtering

In this paragraph, we illustrate the behavior of our approach to perform noisy images filtering. Let $f_0: V \rightarrow \mathbb{R}^n$ be a level set function that represents initial data.

Fig. 3 presents a comparison between the effects of filtering of a noisy image using mean curvature flow on local and nonlocal graph structures. The first line presents results obtained on a local graph. The second line presents results obtained on a nonlocal graph. In this example, we construct a 16-adjacency grid graph with $w(u,v)=e^{(-d(F(u),F(v))/\sigma^2)}$ and $\sigma=20$, where $F: V \times \mathbb{R}^d$ associates a patch of pixels to every vertex (this to better describe texture informations). Both lines present results at different steps of mean curvature flow. Similar to the previous example, one can remark that nonlocal structure better preserves image details.

Fig. 4 presents the filtering of a noisy Lena image noised by white Gaussian noise with different values of *sigma* and a noisy colored Lena image. The graph construction is the same as the example in Fig. 3. One can see that our approach preserves the image details. Figs. 5–9 present a comparison of mean curvature filtering of a textured noisy image on local and nonlocal graph structures. The first line presents results obtained on a local graph, the second one presents results obtained on a nonlocal graph. The graph construction is the same as the example in Fig. 3. One can see that the filtering on nonlocal graph structure better preserves the image textures.

Table 1 presents the comparison of different filtering methods on the Lena noisy image; the methods' psnr values were taken from [23]. In this example, we added a white Gaussian noise with different values of *sigma*. The results of our filtering are shown in Fig. 3 (first and second lines). One can see that our approach has a good value of psnr comparing with the other approaches.

4.4. Application to images on surfaces filtering

In this paragraph, we illustrate the adaptivity of our approach to perform noisy image on surfaces filtering. The approach is the same that for image on regular grid, but with a different graph topology. Fig. 10 presents the filtering of an image on a 3D surface (or mesh) on local and nonlocal graph structures. The first line shows the initial model, the second one shows the filtering result on a local graph structure and the third one shows the filtering result on a nonlocal graph structure. The graph was constructed using the mesh structure.

4.5. Application to 3D surface smoothing

Another application that illustrates the adaptivity of our approach is the 3D surface smoothing. Fig. 11 presents a 3D noisy surface smoothing using mean curvature flow and the fast surface reconstruction method presented in [3]. In this example, the graph is a k-NN graph constructed from the set of initial points and numerous additional points as described in [3]. The weight function is given by $w(u,v)=e^{(-d(f^0(u),f^0(v))/\sigma^2)}$ with $\sigma=10$. Similar to the first example, the level set function f_0 is defined as $f_0=\chi_{\Omega_0}-\chi_{\Omega_0^c}$.

5. Conclusion

In this paper, we have introduced an adaptation of the mean curvature on weighted graphs as the first variation of perimeters, based on PdEs and using a framework of discrete operators. Experimental results have shown the potentiality of the proposed formulation of mean curvature level sets and its adaptivity to graphs of arbitrary topology.

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