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On the p-Laplacian and ∞-Laplacian on Graphs with Applications in Image and Data Processing*

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Abstract. In this paper we introduce a new family of partial difference operators on graphs and study equations involving these operators. This family covers local variational p-Laplacian, ∞-Laplacian, nonlocal p-Laplacian and ∞-Laplacian, p-Laplacian with gradient terms, and gradient operators used in morphology based on the partial differential equation. We analyze a corresponding parabolic equation involving these operators which enables us to interpolate adaptively between p-Laplacian diffusion-based filtering and morphological filtering, i.e., erosion and dilation. Then, we consider the elliptic partial difference equation with its corresponding Dirichlet problem and we prove the existence and uniqueness of respective solutions. For $p = \infty$, we investigate the connection with Tug-of-War games. Finally, we demonstrate the adaptability of this new formulation for different tasks in image and point cloud processing, such as filtering, segmentation, clustering, and inpainting.

Key words. p-Laplacian and ∞-Laplacian, morphological operators, nonlocal PDEs, graphs, Tug-of-War game, image processing, machine learning, inpainting

AMS subject classifications. 35R02, 34B45, 35J92, 49M25, 68U10, 62H30, 91C20

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1. Introduction.

Partial differential equations (PDEs) play a key role for mathematical modeling throughout applied and natural sciences. In this context, the variational p-Laplacian and ∞-Laplacian and its related variant, the game p-Laplacian, represent fundamental differential operators, which have been used to describe many important processes, e.g., in physics, biology, or economy; see [19, 35, 41]. Recently, the nonlocal p-Laplacian has also gained growing interest in the literature, as it appears naturally in the study of nonlocal diffusion processes, as well as in mathematical biology, peridynamics, and image processing [1, 2, 28]. There exist various possibilities to approximate continuous PDEs involving Laplacian formulations in different forms on discrete domains. In the setting of Euclidean domains discretization schemes based on finite differences, finite elements, finite volumes, etc., are well-investigated and traditionally used [57]. On the other hand, important physical processes also arise in more complex geometric environments, e.g., for point cloud data or surfaces. Processing and

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analyzing these types of data is a major challenge and often the discretization of differential operators becomes more difficult compared to the previously mentioned approaches. One can classify possible approaches into implicit methods [6, 42, 48], explicit methods [53, 54], and intrinsic methods [33]. For a more detailed discussion of these methods and their respective advantages see, e.g., [37].

Recently, there is high interest in adapting and solving PDEs on data which is given by arbitrary graphs and networks. The demand for such methods is motivated by existing and potential future applications, such as in machine learning and mathematical image processing. Indeed, any kind of data can be represented by a graph in an abstract form in which the vertices are associated to the data and the edges correspond to relationships within the data. In order to translate and solve PDEs on graphs, different discrete vector calculus have been proposed in the literature in recent years; e.g., see [30] and references therein. One simple discrete calculus on graphs is based on discrete partial differences [23, 24], which enables one to solve PDEs on both regular as well as irregular data domains in a unified and simple manner. This mimetic approach consists of replacing continuous partial differential operators, e.g., gradient or divergence, by a reasonable discrete analogue, which makes it possible to transfer many important tools and results from the continuous setting. This leads to the formulation and solution of partial difference equations (PdEs) on graphs. These PdEs on graphs have been studied as a topic of their own interest and theoretic results such as the existence of respective solutions for the latter have been shown [34, 40, 44]. In particular, studying appropriate formulations of the variational p-Laplacian and ∞-Laplacian on graphs gets more and more into the focus of research as it is well-motivated by many possible applications, e.g., in image processing (denoising, segmentation, inpainting), but also applications in machine learning, such as data processing and data clustering [7, 18, 22, 30, 51].

The main goal in this paper is to introduce a novel class of p-Laplacians and ∞-Laplacians on graphs with gradient terms based on partial difference operators. An interesting feature of the proposed class of operators is the fact that it is able to interpolate adaptively between terms which correspond to nonlocal diffusion-based filters and terms related to nonlocal morphological filter types, i.e., erosion and dilation. Hence, one is able to combine the advantages of both formulations within the same framework. Furthermore, we are able to show that our novel class of p-Laplacians and ∞-Laplacians on graphs with additional gradient terms is able to recover many existing discretization schemes for the local and nonlocal cases and also known graph-based formulations in the literature. In consequence, we are able to give a unified discrete formulation for the p-Laplacian and ∞-Laplacian, the game p-Laplacian, and the nonlocal p-Laplacian on graphs for both regular and irregular discrete domains.

1.1. Contributions. The main contributions of this work are manifold. First, we give a comprehensive overview of the p-Laplacian and ∞-Laplacian and its related variants, the game p-Laplacian and the nonlocal p-Laplacian on Euclidean domains with their respective applications in image and data processing. We discuss how to translate these continuous Laplacian formulations to graphs based on previous works on this topic and how to recover discrete local discretizations traditionally used in image processing. Then, we propose a novel class of p-Laplacians and ∞-Laplacians on graphs which unify many existing discretization schemes, both local and nonlocal. In particular, our proposed formulation can be expressed as a convex interpolation between two discrete upwind gradient terms. An interesting feature of this representation is the fact that the respective steering parameters can be chosen data-dependent, which leads to adaptive filtering effects (nonlocal diffusion and morphological filtering) in different regions of the same data. Subsequently, we discuss the connection to local and nonlocal PDEs and a model from stochastic game theory known as the Tug-of-War game. Again, we show that the proposed unified formulation, using $p = \infty$, leads to PdEs which coincide with value functions of many Tug-of-War games.

We apply this novel class of the p-Laplacian and ∞-Laplacian on two PdEs on weighted graphs which are related to two classical PDEs in the continuous setting, and we prove important mathematical properties, e.g., existence and uniqueness of solutions. First, we investigate a family of parabolic PdEs with initial conditions, leading to a generalization of diffusion and shock filtering on regular and irregular discrete domains. Then, we study a family of elliptic PdEs with Dirichlet boundary conditions generalizing interpolation processes on discrete domains and prove the existence and uniqueness of respective solutions.

Finally, we illustrate how this new class of p-Laplacians and ∞-Laplacians with gradient terms can be applied in many examples, i.e., segmentation, denoising, inpainting, and clustering. To underline the universal applicability of the proposed formulation we test our algorithms on a wide range of data, i.e., classical images, triangulated meshes, and even unorganized data such as point clouds or databases.

1.2. Paper organization.

This paper is organized as follows. We begin by summarizing different Laplacian formulations in the continuous setting of Euclidean spaces in section 2. Subsequently, we give the needed definitions and notation in section 3 to translate Laplacian formulations to graphs and we discuss previous works on this topic. In section 4 we derive a novel class of partial difference operators which unifies many discrete formulations of the p-Laplacian and its mentioned variants. We show the connection between the proposed operator and local and nonlocal PDEs in continuous and discrete settings. We also show that this operator is related to different version of the stochastic Tug-of-War game. We apply the proposed formulation on two families of PdEs related to classical PDEs in the continuous setting in section 5 and give the respective analysis for these problems. Section 6 presents several applications, such as denoising or segmentation, on regular images and high-dimensional unorganized data. Finally, we conclude this paper by a short discussion in section 7.

2. p-Laplacian formulations on Euclidean spaces.

We give a comprehensive overview of continuous Laplacian formulations on Euclidean domains in the following. Since this is the base of our work we review previous works on this topic and thus make this paper more self-contained. Additionally, we discuss applications in image and data processing and give links to other mathematical fields. We start by a discussion of the variational p-Laplacian and ∞-Laplacian operators in section 2.1. Subsequently, we introduce the so-called game p-Laplacian in section 2.2 and investigate the relationship to the latter variational formulation and the stochastic Tug-of-War game. Finally, we give the definition of the nonlocal p-Laplacian in section 2.3.

In the following we denote by $\Omega \subset \mathbb{R}^n$ an open, bounded domain and $u\colon \Omega \to \mathbb{R}$ a function on Ω.

2.1. The variational p-Laplacian and ∞-Laplacian. The variational p-Laplacian plays an important role in continuous geometry but also in the field of PDEs, which are used to describe many phenomena in physics or biology. For an introduction and survey on this topic see, e.g., [19, 35] and references therein. The variational p-Laplace operator is a quasi-linear elliptic partial differential operator of second order and can be formulated as

$$\Delta_p u \;=\; \operatorname{div}\left(\frac{\nabla u}{|\nabla u|^{2-p}}\right) \;, \quad 1 \leq p < \infty \;. \tag{2.1}$$

It arises from the Euler–Lagrange equation for minimization of the functional

$$E(u) \;=\; \frac{1}{p}\int_{\Omega} |\nabla u(x)|^p \,\mathrm{d}x \;, \tag{2.2}$$

leading to the PDE

$$\Delta_p u \;=\; 0 \tag{2.3}$$

with Dirichlet boundary conditions. Many inverse problems in image processing such as denoising, deconvolution, segmentation, or inpainting are formulated with the help of regularization terms based on weak formulations of the p-Laplacian in (2.1). Note that for $p = 2$ one retrieves the classical Laplace operator Δ. The variational p-Laplacian is linked to the well-known Thikonov regularization (for the case $p = 2$) and the total variation regularization (for the case $p = 1$).

For $p = \infty$, the $\infty-$Laplacian is defined as [4]

$$\Delta_\infty u \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i x_j} \;. \tag{2.4}$$

2.2. The game p-Laplacian. Recently, a variant of the p-Laplacian known as game p-Laplacian has been introduced in connection with a stochastic game called Tug-of-War with noise [47]. The game p-Laplacian is based on the variational p-Laplacian in (2.1) and can be formulated as

$$\Delta_p^G u \;=\; \frac{1}{p}|\nabla u|^{2-p}\Delta_p u \;=\; \frac{1}{p}|\nabla u|^{2-p}\operatorname{div}\left(\frac{\nabla u}{|\nabla u|^{2-p}}\right) \;, \quad 1 \leq p < \infty \;. \tag{2.5}$$

Following [46], a definition of the game $\infty-$Laplacian is given by

$$\Delta_\infty^G u \;=\; |\nabla u|^{-2}\Delta_\infty u \;. \tag{2.6}$$

It gets clear that one has the following relationships:

$$\begin{aligned} \Delta_1^G u \;&=\; \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)|\nabla u| \;, \\ \Delta_2^G u \;&=\; \frac{1}{2}\Delta u \;. \end{aligned} \tag{2.7}$$

If f is a smooth function, then (2.5) can be rewritten as a convex combination of the 2-Laplacian and the game ∞-Laplacian in (2.6) as follows:

$$\Delta_p^G u \;=\; a\Delta_2^G u \,+\, b\Delta_\infty^G u \qquad \text{for } a = \frac{2}{p},\ b = \frac{p-2}{p}\ . \tag{2.8}$$

Since $\Delta_1^G u = \Delta_2 u - \Delta_\infty^G u$ one can rewrite (2.5) as

$$\Delta_p^G u \;=\; a\Delta_2^G u \,+\, b\Delta_1^G u \qquad \text{for } a = \frac{2(p-1)}{p},\ b = \frac{2-p}{p}\ . \tag{2.9}$$

Another known relationship is given by

$$\Delta_p^G u \;=\; \frac{1}{p}\Delta_1^G u \,+\, \frac{1}{q}\Delta_\infty^G u \qquad \text{for } \frac{1}{p} + \frac{1}{q} = 1 \ \text{ and } \ 1 < p, q < \infty\ . \tag{2.10}$$

The game p-Laplacian is also called normalized as it is homogeneous of degree 1, i.e., for $a \in \mathbb{R}$ one has $\Delta_p^G(au) = a\Delta_p^G(u)$, in contrast to the variational p-Laplacian in (2.1), which is homogeneous of degree $(p-1)$. This leads to the fact that parabolic PDEs involving the game p-Laplacian are scaling invariant, which is especially useful for many applications in mathematical image processing.

For $p = 1$ this operator is closely related to the mean curvature flow, which has numerous applications ranging from free boundary problems in material sciences and computational fluid dynamic to filtering, inpainting, and segmentation in image processing and computer vision; e.g., see [50] and references therein. For formally $p = \infty$ the game p-Laplacian has been used in several applications in image processing, computer vision, surface reconstruction, and image inpainting [22, 26].

Recently, a link has been shown between the Tug-of-War game and the game p-Laplacian for $p = \infty$ [46], and the Tug-of-War with noise and the game p-Laplacian for $1 < p < \infty$ [38, 47].

2.3. The nonlocal p-Laplacian. The interest in the fractional and nonlocal Laplacian has constantly increased over the last few years. These operators are used in various applications such as continuum mechanics, phase transition phenomena, population dynamics, image processing, and game theory; see [2, 1]. In image processing, regularization based on the nonlocal p-Laplacian [27] is related to works on nonlocal image processing such as initially proposed by Buades, Coll, and Morel [12]. Nonlocal regularization methods have shown great advantages over classical models in certain applications, since local smoothness is not required. They have also shown their ability to preserve both geometric and repetitive structures in images. For $1 \leq p < \infty$, the nonlocal p-Laplacian is defined as follows [1]:

$$\mathcal{L}_p u(x) \;=\; \int_\Omega \mu(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,\mathrm{d}y\ , \quad 1 \leq p < \infty\ . \tag{2.11}$$

In this case $\mu\colon \mathbb{R}^n \to \mathbb{R}$ is a nonnegative continuous radial function with compact support and $\mu(0) > 0$ and $\int_{\mathbb{R}^n} \mu(x)\mathrm{d}x = 1$. The evolution equation involving this operator has been studied in [1]. The authors of [1] also show that this operator is derived from the Euler–Lagrange equation for minimization of the following energy:

$$J_p(u) \;= \frac{1}{2p} \int_\Omega \int_\Omega \mu(x-y)|u(y)-u(x)|^p \,\mathrm{d}x\,\mathrm{d}y \;. \tag{2.12}$$

In particular, for

$$\mu(x,y) = \frac{1}{|x-y|^{\alpha p}}, \tag{2.13}$$

with $\alpha = n/p + s$, $0 < s < 1$, and $p \geq 1$, we recover the fractional p-Laplacian:

$$\mathcal{L}_p u(x) \;=\; \int_\Omega \frac{1}{|x-y|^{\alpha p}} |u(y)-u(x)|^{p-2}(u(y)-u(x))\,\mathrm{d}y. \tag{2.14}$$

By using (2.13) and $p = \infty$ one can recover the Hölder infinity Laplacian [15],

$$\mathcal{L}_\infty u(x) \;=\; \max_{y\in\Omega, y\neq x} \left(\frac{u(y)-u(x)}{|y-x|^\alpha} \right) + \min_{y\in\Omega, y\neq x} \left(\frac{u(y)-u(x)}{|y-x|^\alpha} \right), \tag{2.15}$$

which can be formally derived as the limit of $p \to \infty$ for the minimization of the following family of energies:

$$J_p(u) \;=\; \int_\Omega \int_\Omega \frac{|u(y)-u(x)|^p}{|x-y|^{\alpha p}} \,\mathrm{d}x\,\mathrm{d}y \;. \tag{2.16}$$

3. Partial differences and the p-Laplacian on graphs.

After the comprehensive overview on continuous formulations of the variational p-Laplacian and its variants in section 2 we discuss in the following how to translate the latter to the discrete setting of graphs. By this we summarize previous works (including our own) on this topic. We begin in section 3.1 by introducing the basic notation and assumptions we need for translating differential operators and PDEs to weighted undirected graphs. Subsequently, we give in section 3.2 the fundamental definitions for difference operators on weighted graphs in order to define derivatives and morphological operators. Based on these, we are able to introduce a formulation of the p-Laplacian and its variants on graphs in section 3.3.

3.1. Basic notation.

A *weighted graph* $G = (V, E, w)$ consists of a finite set V of $N \in \mathbb{N}$ vertices, a finite set $E \subseteq V \times V$ of edges, and a weight function $w : V \times V \to [0,1]$. In our case the weight function represents a similarity measure between two vertices of the graph. We denote by $(u,v) \in E$ the edge that connects the vertices u and v and we write $u \sim v$ for two adjacent vertices. The *neighborhood* of a vertex u (i.e., the set of vertices adjacent to u) is denoted by $N(u)$ and the *degree* of a vertex u is defined as $\delta_w(u) = \sum_{v\sim u} w(u,v)$. For two vertices $u, v \in V$ with $u \nsim v$ we set $w(u,v) = w(v,u) = 0$ and thus the set of edges E can be characterized by the weight function w as $E = \{(u,v) \mid w(u,v) > 0\}$. A weighted graph G is called *undirected* if for every $u, v \in V$ the weight function w satisfies the symmetry condition $w(u,v) = w(v,u)$.

Let $\mathcal{H}(V)$ be the Hilbert space of real valued functions on the vertices of the graph, i.e., each function $f : V \to \mathbb{R}$ in $\mathcal{H}(V)$ assigns a real value $f(u)$ to each vertex $u \in V$. For a function $f \in \mathcal{H}(V)$ the $\mathcal{L}^p(V)$ norm of f is given by

$$\begin{aligned} \|f\|_p &= \Big(\sum_{u\in V} |f(u)|^p\Big)^{1/p} && \text{for } 1 \leqslant p < \infty \ , \\ \|f\|_\infty &= \max_{u\in V}\big(|f(u)|\big) && \text{for } p = \infty \ . \end{aligned} \tag{3.1}$$

The Hilbert space $\mathcal{H}(V)$ is endowed with the following inner product: $\langle f, g\rangle_{\mathcal{H}(V)} = \sum_{u\in V} f(u)\, g(u)$ with $f, g \in \mathcal{H}(V)$. Similarly, let $\mathcal{H}(E)$ be the Hilbert space of real valued functions defined on the edges of the graph, i.e., each function $F : E \to \mathbb{R}$ in $\mathcal{H}(E)$ assigns a real value $F(u, v)$ to each edge $(u, v) \in E$. The Hilbert space $\mathcal{H}(E)$ is then endowed with the following inner product: $\langle F, G\rangle_{\mathcal{H}(E)} = \sum_{u\in V}\sum_{v\in V} F(u, v)G(u, v)$ for $F, G \in \mathcal{H}(E)$.

Let $\mathcal{A} \subset V$ be a set of connected vertices, i.e., $\forall\, u \in \mathcal{A}$ there exists a vertex $v \in \mathcal{A}$ with $(u, v) \in E$. We denote by $\partial\mathcal{A}$ the *(outer) boundary set* of $\mathcal{A}$, which is given by

$$\partial\mathcal{A} \ = \ \{u \in \mathcal{A}^c : \exists v \in \mathcal{A} \text{ with } (u, v) \in E\} \ , \tag{3.2}$$

where $\mathcal{A}^c = V \setminus \mathcal{A}$ is the complementary set of $\mathcal{A}$ in V.

3.2. Weighted partial differences on graphs. Using the basic notation given in section 3.1 we are able to introduce the needed framework to translate differential operators and PDEs from the continuous setting to graphs. In particular the fundamental elements for this translation are weighted partial differences on graphs. For more detailed information on these operators we refer to [23, 9, 56]. In the following we assume that the considered graphs are connected and undirected, with neither self-loops nor multiple edges between vertices.

Let $G = (V, E, w)$ be a weighted graph and let $f \in \mathcal{H}(V)$ be a function on the set of vertices V of G. Then we can define the *weighted partial difference* of f at a vertex $u \in V$ in direction of a vertex $v \in V$ as

$$\partial_v f(u) \ = \ \sqrt{w(u, v)}\,(f(v) - f(u)) \ . \tag{3.3}$$

As for the continuous definition of directional derivatives, we have the following properties $\partial_v f(u) = -\partial_u f(v)$, $\partial_u f(u) = 0$, and if $f(u) = f(v)$, then $\partial_v f(u) = 0$.

Based on the definition of weighted partial differences in (3.3) one can straightforwardly introduce the *weighted gradient operator* on graphs $\nabla_w : \mathcal{H}(V) \to \mathcal{H}(E)$, which is defined on a vertex $u \in V$ as the vector of all weighted finite differences with respect to the set of vertices V, i.e.,

$$(\nabla_w f)(u) \ = \ (\partial_v f(u))_{v\in V} \, . \tag{3.4}$$

From the properties of the weighted partial differences above it gets clear that the weighted gradient is linear and antisymmetric. The weighted gradient at a vertex $u \in V$ can be interpreted as a function in $\mathcal{H}(V)$ and hence the $\mathcal{L}^p(V)$ and $\mathcal{L}^\infty(V)$ norm in (3.1) of this finite vector represent its respective *local variation* and are given as

$$\begin{aligned} \|(\nabla_w f)(u)\|_p &= \left(\sum_{v\sim u} \sqrt{(w(u,v))}^p \, |f(v)-f(u)|^p \right)^{\frac{1}{p}} , \\ \|(\nabla_w f)(u)\|_\infty &= \max_{v\sim u} \left(\sqrt{w(u,v)} \, |f(v)-f(u)| \right) . \end{aligned} \tag{3.5}$$

The *difference operator* of a function $f \in \mathcal{H}(V)$, noted $\mathcal{G}_w : \mathcal{H}(V) \to \mathcal{H}(V \times V)$, is defined on a pair of vertices $(u,v) \in E$ by

$$(\mathcal{G}_w f)(u,v) \;=\; \sqrt{w(u,v)}\,(f(v)-f(u)) . \tag{3.6}$$

This operator is linear and antisymmetric.

The *adjoint operator* of the difference operator $\mathcal{G}_w^* : \mathcal{H}(E) \to \mathcal{H}(V)$ is a linear operator defined by $\langle \mathcal{G}_w f, H\rangle_{\mathcal{H}(E)} = \langle f, \mathcal{G}_w^* H\rangle_{\mathcal{H}(V)}$ $\forall\, f \in \mathcal{H}(V)$ and $\forall\, H \in \mathcal{H}(E)$. Using the definitions of difference and inner products in $\mathcal{H}(V)$ and $\mathcal{H}(E)$, the adjoint operator $\mathcal{G}_w^*$, of a function $H \in \mathcal{H}(E)$, can be expressed at a vertex $u \in V$ by the following expression:

$$(\mathcal{G}_w^* H)(u) \;=\; \sum_{v\sim u} \sqrt{w(u,v)}(H(v,u) - H(u,v)). \tag{3.7}$$

The *divergence operator*, defined by

$$D_w = -\mathcal{G}_w^*, \tag{3.8}$$

measures the net outflow of a function of $\mathcal{H}(E)$ at each vertex of the graph. Each function $H \in \mathcal{H}(E)$ has a null divergence over the entire set of vertices. From the previous definitions, it can be easily shown that $\sum_{u\in V}\sum_{v\in V} \mathcal{G}_w f(u,v) = 0, f \in \mathcal{H}(v)$, and $\sum_{u\in V} D_w F(u) = 0, F \in \mathcal{H}(E)$.

Based on the previous definitions we can define two *upwind directional derivatives* expressed by

$$\partial_v^{\pm} f(u) = \sqrt{w(u,v)}\big(f(v)-f(u)\big)^{\pm} \tag{3.9}$$

with the notation $(x)^+ = \max(0,x)$ and $(x)^- = -\min(0,x) = \max(0,-x)$.

Similarly, the *upwind weighted gradient* can be defined as

$$(\nabla_w^{\pm} f)(u) \;=\; \left(\partial_v^{\pm} f(u)\right)_{v\in V} . \tag{3.10}$$

The *upwind gradient norm* with $1 \leqslant p < \infty$ is defined for a function $f \in \mathcal{H}(V)$ as

$$\|\nabla_w^{\pm} f(u)\|_p \;=\; \left[\sum_{v\sim u} \sqrt{w(u,v)}^p \big(f(v)-f(u)\big)^{p\pm} \right]^{\frac{1}{p}} . \tag{3.11}$$

This operator measures the regularity of a function around a vertex u. In the case $p = \infty$ one gets

$$\|\nabla_w^{\pm} f(u)\|_\infty \;=\; \max_{v\sim u}\Big(\sqrt{w(u,v)}\big(f(v)-f(u)\big)^{\pm}\Big)\;. \tag{3.12}$$

A useful relationship between the weighted gradient and its upwind variant is given for a function $f \in \mathcal{H}(V)$ by

$$\|\nabla_w f(u)\|_p^p \;=\; \|\nabla_w^{+} f(u)\|_p^p \;+\; \|\nabla_w^{-} f(u)\|_p^p\;, \tag{3.13}$$

and one can deduce that

$$\|\nabla_w^{\pm} f(u)\|_p \;\leqslant\; \|\nabla_w f(u)\|_p\;. \tag{3.14}$$

Thus the family of upwind gradients provides a slightly finer expression of the gradient. For instance, one can remark that $\|\nabla_w^{-} f(u)\|_p$ is always zero if f has a local minimum at u. The upwind gradient was used in [18, 55] to adapt the Eikonal equation to weighted graphs and to study existence and uniqueness of respective solutions with applications in image processing and machine learning.

Finally, the family of gradient operators introduced above can be used to construct several (nonlocal) regularization functionals on graphs. For instance,

$$\begin{aligned}
J_{p,w}(f) &= \sum_{u\in V} \|\nabla_w f(u)\|_p^p\;, \quad 1 \leqslant p < \infty\;,\\
J_{\infty,w}(f) &= \sum_{u\in V} \|\nabla_w f(u)\|_\infty\;,\\
J_{p,w}^{\pm}(f) &= \sum_{u\in V} \|\nabla_w^{\pm} f(u)\|_p^p\;, \quad 1 \leqslant p < \infty\;,\\
J_{\infty,w}^{\pm}(f) &= \sum_{u\in V} \|\nabla_w^{\pm} f(u)\|_\infty\;.
\end{aligned} \tag{3.15}$$

3.3. Our previous works on p-Laplacian on graphs. The graph p-Laplacian, a generalization of the discrete p-Laplacian, started to attract attention in mathematics, machine learning, and in the image and manifold processing communities. For $p \neq 2$ the graph p-Laplacian has been studied in relation with the p-cheeger cut and data clustering [31] and for semisupervised classification [59]. Meanwhile, PdEs on graphs based on the discrete p-Laplacian have been investigated as a subject of their own interest, dealing with existence and qualitative behavior of respective solutions [34, 40, 44]. In previous works, we have introduced a nonlocal discrete vector calculus to translate many PDEs and variational methods to graphs. In [9, 23] we have introduced nonlocal regularization on weighted graphs of arbitrary topology. In particular, it was shown that these regularizations lead to a family of discrete and semidiscrete diffusion processes based on the discrete p-Laplacian. These processes, parametrized by the graph structure (topology and geometry) and by the degree p of smoothness, allow us to perform several filtering tasks such as denoising, simplification, or clustering. Moreover, local and nonlocal image regularization are formalized within the same framework, which corresponds to the transcription of local or nonlocal regularization proposed in [28]. With the same ideas, we have proposed PdE-based morphological processes on graphs to transcribe continuous morphological PDEs such as dilation or erosion [56]. The study of well posedness of the Eikonal equation on graphs was proposed in [18]. Recently we have also proposed the adaptation of both nonlocal infinity Laplacian [21] and game p-Laplacian for $2 \leq p \leq \infty$ on graphs [22].

In graph theory, there are different expressions for the p-Laplacian on graphs [13, 34]. In the context of PdEs on graphs, based on the weighted partial differences (see (3.6)) and the divergence operators (see (3.8)), we mimic the classical definition of the p-Laplacian on Euclidean domains to derive a unified form for two expressions: anisotropic and isotropic p-Laplacian.

Anisotropic graph p-Laplacian. The *anisotropic graph p-Laplacian* of a function $f \in \mathcal{H}(V)$, denoted by $\Delta^a_{w,p} : \mathcal{H}(V) \to \mathcal{H}(V)$, is defined as

$$(\Delta^a_{w,p} f)(u) \;=\; \tfrac{1}{2} D_w\Big(|G_w f|^{p-2} G_w f\Big)(u) \qquad \text{for } 1 \leq p < \infty\ . \tag{3.16}$$

Using (3.6) and (3.7), the anisotropic p-Laplacian of $f \in \mathcal{H}(V)$ at a vertex $u \in V$ can be computed as [9, 23]

$$(\Delta^a_{w,p} f)(u) \;=\; \sum_{v \sim u} \sqrt{w(u,v)}^{\,p} |f(v) - f(u)|^{p-2} (f(v) - f(u))\ . \tag{3.17}$$

Remark. As in the continuous case, this operator can be formally derived from minimization of the following energy on graphs:

$$J_{w,p}(f) \;=\; \frac{1}{2p} \sum_{u \in V} \|\nabla_w f(u)\|_p^p\ . \tag{3.18}$$

Isotropic graph p-Laplacian. The *isotropic graph p-Laplacian* noted $\Delta^i_{w,p} : \mathcal{H}(V) \to \mathcal{H}(V)$ is defined by

$$(\Delta^i_{w,p} f)(u) \;=\; \tfrac{1}{2} D_w\Big(\|\nabla_w f\|_2^{p-2} G_w f\Big)(u) \qquad \text{for } 1 \leq p < \infty\ . \tag{3.19}$$

Using (3.6) and (3.7), the isotropic p-Laplacian of $f \in \mathcal{H}(V)$, at a vertex $u \in V$, can be computed by

$$(\Delta^i_{w,p} f)(u) \;=\; \frac{1}{2} \sum_{v \in V} w(u,v) \big(\|\nabla f(v)\|_2^{p-2} + \|\nabla f(u)\|_2^{p-2}\big) \big(f(v) - f(u)\big)\ . \tag{3.20}$$

Similarly to the anisotropic case, this form of the p-Laplacian can be interpreted as the first variation of the following energy:

$$J_{w,p}(f) \;=\; \frac{1}{p} \sum_{u \in V} \|\nabla_w f(u)\|_2^p\ . \tag{3.21}$$

For $p = 2$ we obtain the classical unnormalized Laplacian for both isotropic and anisotropic Laplacian as

$$(\Delta^u_{w,2} f)(u) \;=\; \sum_{v \sim u} w(u,v)\,(f(v) - f(u))\,. \tag{3.22}$$

Infinity Laplacian. The nonlocal infinity Laplacian of a function $f \in \mathcal{H}(V)$, denoted $\Delta_{w,\infty} : \mathcal{H}(V) \to \mathcal{H}(V)$, is defined by [20]

$$\Delta_{w,\infty} f(u) \;=\; \frac{1}{2}\left[\|\nabla_w^+ f(u)\|_\infty - \|\nabla_w^- f(u)\|_\infty\right] \;, \tag{3.23}$$

which can be rewritten as

$$\begin{aligned}\Delta_{w,\infty} f(u) \;=\; &\frac{1}{2}\Big[\max\Big(\sqrt{w(u,v)}\big(f(v)-f(u)\big)^+\Big) \\ &-\max\Big(\sqrt{w(u,v)}\big(f(v)-f(u)\big)^-\Big)\Big] \;.\end{aligned} \tag{3.24}$$

Remark. As in the continuous case, this operator can be formally derived as minimization of the following family of energies on graphs in the limit for $p \to \infty$:

$$J_{w,p}(f) \;=\; \sum_{u\in V} \|\nabla_w f(u)\|_p \;. \tag{3.25}$$

For more details, see [9, 23]

Normalized p-Laplacian. In [21] we proposed and studied a discretization of the normalized p-Laplacian on weighted graphs, for $p \geq 2$, using the identity (2.8). To achieve this we proposed a graph version of the normalized 2-Laplacian:

$$\Delta_{w,2}^G f(u) \;=\; \frac{\sum_{v\sim u} w(u,v) f(v)}{\delta_w(u)} - f(u) \;. \tag{3.26}$$

Then, the discrete normalized p-Laplacian of a function $f \in \mathcal{H}(V)$, denoted by $\Delta_{\alpha,\beta} : \mathcal{H}(V) \to \mathcal{H}(V)$, is defined in [22] as

$$\Delta_{\alpha,\beta} f \;=\; \tfrac{\alpha}{2}\left[\|\nabla_w^+ f\|_\infty - \|\nabla_w^- f\|_\infty\right] + \beta \Delta_{w,2}^G f \;. \tag{3.27}$$

Remark. All the presented operators can have either local or nonlocal effects, depending on the graph topology.

4. A novel class of graph p-Laplacians and ∞-Laplacians with gradient terms. In this section we propose a novel discrete operator on weighted graphs that corresponds to a new class of p-Laplace operators with gradients terms. We show that our proposed operator leads to a general partial difference operator, unifying the operators on graphs presented in section 3 and extending them in a formulation that interpolates between p-Laplacian and morphological operators on graphs. In section 4.1, we begin by defining the proposed operator, and we show that particular cases of this operator enable us to recover either the p-Laplacian or the morphological operator we previously defined. Then we study some connections between this family and local and nonlocal differential operators in section 4.2 and Tug-of-War games in section 4.3.

4.1. Definition of the novel operator. Using the discretization of p-Laplacian and infinity Laplacian on a general graph domain as introduced in section 3.3, we now introduce a new expression for both these operators.

Definition 4.1. *The $\mathcal{L}_{w,p}$ and $\mathcal{L}_{w,\infty}$ Laplacians are defined for a function $f \in \mathcal{H}(V)$ by*

$$\mathcal{L}_{w,p}f(u) = \begin{cases} \alpha(u)\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} - \beta(u)\|(\nabla_w^- f)(u)\|_{p-1}^{p-1}, & 2 \le p < \infty, \\ \alpha(u)\|(\nabla_w^+ f)(u)\|_\infty - \beta(u)\|(\nabla_w^- f)(u)\|_\infty, & p = \infty, \end{cases} \tag{4.1}$$

with $\alpha(u), \beta(u) : \mathcal{H}(V) \to [0,1]$, and $\alpha(u) + \beta(u) = 1$.

By a simple factorization these operators can be rewritten as

$$\begin{aligned} \mathcal{L}_{w,p}f(u) = {} & 2\min(\alpha(u),\beta(u))\Delta_{w,p}f(u) \\ & + (\alpha(u)-\beta(u))^+\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} \\ & - (\alpha(u)-\beta(u))^-\|(\nabla_w^- f)(u)\|_{p-1}^{p-1}, \quad 2 \le p < \infty, \\ \mathcal{L}_{w,\infty}f(u) = {} & 2\min(\alpha(u),\beta(u))\Delta_{w,\infty}f(u) \\ & + (\alpha(u)-\beta(u))^+\|(\nabla_w^+ f)(u)\|_\infty \\ & - (\alpha(u)-\beta(u))^-\|(\nabla_w^- f)(u)\|_\infty, \qquad p = \infty. \end{aligned} \tag{4.2}$$

Considering constant functions $\alpha(u) = \alpha$ and $\beta(u) = \beta$, this expression recovers well-known expressions of Laplacian, infinity Laplacian, or p-Laplacian on graphs and their operators with gradient terms depending on the choice of α, β:

- In the case $\alpha = \beta \neq 0$ the operator in (4.2) becomes

$$\begin{aligned} \mathcal{L}_{w,p}f(u) &= \Delta_{w,p}f(u), \\ \mathcal{L}_{w,\infty}f(u) &= \Delta_{w,\infty}f(u) \end{aligned} \tag{4.3}$$

  and thus recovers the discrete p-Laplacian and ∞-Laplacian expressions.
- In the case $\alpha = 1$ the operator in (4.2) becomes

$$\begin{aligned} \mathcal{L}_{w,p}f(u) &= \|(\nabla_w^+ f)(u)\|_{p-1}^{p-1}, \\ \mathcal{L}_{w,\infty}f(u) &= \|(\nabla_w^+ f)(u)\|_\infty, \end{aligned} \tag{4.4}$$

  and for $\beta = 1$ it becomes

$$\begin{aligned} \mathcal{L}_{w,p}f(u) &= -\|(\nabla_w^- f)(u)\|_{p-1}^{p-1}, \\ \mathcal{L}_{w,\infty}f(u) &= -\|(\nabla_w^- f)(u)\|_\infty. \end{aligned} \tag{4.5}$$

  We can see that we recover PdE-based morphological operators with the upwind gradient discretization [56].
- In the case $\alpha - \beta > 0$ the operator in (4.2) becomes

$$\begin{aligned} \mathcal{L}_{w,p}f(u) &= 2\beta(u)\Delta_{w,p}f(u) \\ & \quad + (\alpha(u)-\beta(u))\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1}, \\ \mathcal{L}_{w,\infty}f(u) &= 2\beta(u)\Delta_{w,\infty}f(u) \\ & \quad + (\alpha(u)-\beta(u))\|(\nabla_w^+ f)(u)\|_\infty. \end{aligned} \tag{4.6}$$

- In the case $\alpha - \beta < 0$ the operator in (4.2) becomes

$$\begin{aligned} \mathcal{L}_{w,p} f(u) &= 2\alpha(u)\Delta_{w,p} f(u) \\ &\quad - (\beta(u) - \alpha(u))\|(\nabla_w^- f)(u)\|_{p-1}^{p-1} , \\ \mathcal{L}_{w,\infty} f(u) &= 2\alpha(u)\Delta_{w,\infty} f(u) \\ &\quad - (\beta(u) - \alpha(u))\|(\nabla_w^- f)(u)\|_{\infty} . \end{aligned} \tag{4.7}$$

Note that for both of the last cases, by using the proposed operator in a parabolic PDE (e.g., $\partial_t f(u) = \mathcal{L}_{w,p} f(u)$), we recover PDE-based operators that are a linear combination between nonlocal diffusion/averaging and shock filtering.

4.2. Connection with discretizations of local and nonlocal differential operators. In this section, we show that our newly introduced p-Laplace operator with gradient terms enables us to recover classical discretization schemes proposed in the literature to solve local and nonlocal PDEs.

For this section we consider Ω an open and bounded domain in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a given function.

4.2.1. Discretizations of local differential operators. Here we first consider the anisotropic p-Laplacian and show that its discretization using second order central differences is recovered by the proposed operator on graphs. Then we investigate the infinity Laplacian and show that the Oberman discretization is also related to our operator. We also discuss different gradient norm discretizations and again show the connection with the proposed operator.

Anisotropic p-Laplacian. The anisotropic p-Laplacian is expressed by

$$\Delta_p^a f \;=\; \sum_{i=1}^{\mathrm{n}} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial f}{\partial x_i}\right|^{p-2} \frac{\partial f}{\partial x_i}\right] . \tag{4.8}$$

If we discretize this expression with second order central differences of the form

$$\frac{\partial}{\partial x_i} f(x) \;=\; D_i(f)(x) \;\approx\; \frac{f(x + h_i/2) - f(x - h_i/2)}{h_i} , \tag{4.9}$$

we get the following discretization of the anisotropic p-Laplacian:

$$\begin{aligned} \Delta_p^a f(x) \;=\; \sum_{i=1}^{n} \frac{1}{h_i^p} \Big(&|f(x_i + h_i) - f(x_i)|^{p-2}(f(x_i + h_i) - f(x_i)) \\ &+ |f(x_i - h_i) - f(x_i)|^{p-2}(f(x_i - h_i) - f(x_i)) \Big) . \end{aligned} \tag{4.10}$$

Let $G(V, E, w)$ be a weighted graph that represents an n-dimensional grid. Let u be a vertex associated to an n-dimensional vector with the spatial coordinates: $u = (i_1 h_1, ..., i_n h_n)^T$, where $i_j \in \mathbb{N}$ and h_j is the grid spacing size with $j = 1, \dots, n$. The neighborhood of u can be defined as $N(u) = \{v : v = u \pm h_j e_j\}_{j=1,\dots,n}$ where $e_j = (q_k)_{k=1,\dots,n}^T$ is the vector such that $q_k = 1$ if $j = k$ and $q_k = 0$ otherwise.

We consider the case where $\alpha(u) = \beta(u) = \frac{1}{2}$ for each $u \in V$, and $2 \leq p < \infty$. Identifying vertices of the graph by their spatial coordinates and setting $w_1(u, v_i) = \frac{1}{h_i^2}$ we get the following formulation:

$$\begin{aligned}
\mathcal{L}_{w_1,p} f(u) \;=\; \Delta^a_{w_1,p} f(u) \;&=\; \sum_{v\sim u} \sqrt{w_1(u,v)}^p |f(v) - f(u)|^{p-2}(f(v) - f(u)) \\
&= \sum_{i=1}^{n} \frac{1}{h_i^p} \big(|f(v_i^+) - f(u)|^{p-2}(f(v_i^+) - f(u)) \\
&\quad + |f(v_i^-) - f(u)|^{p-2}(f(v_i^-) - f(u))\big)
\end{aligned} \tag{4.11}$$

with $v_i^{\pm} = u \pm h_i e_i$. One can see that we recover the discrete formulation of the local anisotropic p-Laplacian in (4.10).

Infinity Laplacian. Let $f(x)$ be a smooth function with nonvanishing gradient at x. Then for the case $p = \infty$ we can mimic the Obermann discretization of the infinity Laplace equation [41] as follows:

$$\Delta_\infty f(x) \;=\; \min_{|y-x|=\epsilon} \frac{(f(y) - f(x))}{\epsilon^2} + \max_{|y-x|=\epsilon} \frac{(f(y) - f(x))}{\epsilon^2} + O(\epsilon^2)\ . \tag{4.12}$$

We use the same graph as discussed above but set the weighting function $w_2(u, v)$ as

$$w_2(u,v) \;=\; \begin{cases} \frac{4}{\epsilon^4} & \text{if } v \in \partial B_\epsilon(u), \\ 0 & \text{otherwise}\ . \end{cases} \tag{4.13}$$

Thus we get

$$\begin{aligned}
\mathcal{L}_{w_2,\infty} f(u) \;&=\; \frac{1}{2}[\max_{v\sim u}(\sqrt{w_2(u,v)}(f(v) - f(u))^+) - \max_{v\sim u}(\sqrt{w_2(u,v)}(f(v) - f(u))^-)] \\
&= \frac{1}{\epsilon^2}[\max_{v\sim u}(f(v) - f(u))^+ - \max_{v\sim u}(f(v) - f(u))^-].
\end{aligned} \tag{4.14}$$

If we define the neighborhood of u as $N(u) \cup \{u\}$, we finally recover the Obermann discretization in (4.12),

$$\begin{aligned}
\mathcal{L}_{w_2,\infty} f(u) \;&=\; \frac{1}{\epsilon^2}[\max_{v\sim u}(f(v) - f(u)) \;-\; \max_{v\sim u}(f(u) - f(v))] \\
&= \frac{1}{\epsilon^2}[\max_{v\sim u}(f(v) - f(u)) \;+\; \min_{v\sim u}(f(v) - f(u))] \\
&= \max_{v\sim u} \frac{f(v) - f(u)}{\epsilon^2} \;+\; \min_{v\sim u} \frac{f(v) - f(u)}{\epsilon^2}\ .
\end{aligned} \tag{4.15}$$

Gradient discretization schemes. If we use the same graph and weighting function w_1 as for the local anisotropic p-Laplacian discussed above and use it for a discretization of the $\mathcal{L}^2$-norm of the upwind gradient, we get

$$\begin{aligned}
\|(\nabla^+_{w_1} f)(u)\|_2^2 \;&=\; \sum_{v\sim u} \sqrt{w_1(u,v)}^2 ((f(v) - f(u))^+)^2 \\
&= \sum_{i=1}^{n} \max(D_i^+ f(u), 0)^2 + \min(D_i^- f(u), 0)^2\ ,
\end{aligned} \tag{4.16}$$

for which D_i^+ and D_i^- denote the classical forward and backward differences:

$$D_i^+(f)(u) = \frac{f(v_i^+) - f(u)}{h_i}, \tag{4.17}$$

$$D_i^-(f)(u) = \frac{f(u) - f(v_i^-)}{h_i}. \tag{4.18}$$

One can see that (4.16) recovers the Osher–Sethian upwind discretization scheme in [43] for $p = 2$. For the case $p = \infty$ we get

$$\begin{aligned} \|(\nabla_{w_1}^+ f)(u,t)\|_\infty &= \max_{v \sim u}(\sqrt{w_1(u,v)}(f(v,t) - f(u,t))^+) \\ &= \max_{i=1,\dots,n}\left(D_i^+ f(u,t), -D_i^- f(u,t), 0\right) , \end{aligned} \tag{4.19}$$

which corresponds in this case to the Godunov discretization scheme for $|(\nabla f)(x)|_\infty$.

4.2.2. Approximations of nonlocal differential operators. In the following, we analyze the connection of the proposed operator to discretizations of nonlocal differentials operators, by using a nonlocal graph construction and a proper weight function: Given a Euclidean graph $G(V, E, w)$, with $V = \Omega \subset \mathbb{R}^n$, $E = \{(x, y) \in V \times V \mid w_3(x, y) > 0\}$, $2 \leq p < \infty$, and

$$w_3(x,y) = \begin{cases} \frac{1}{|x-y|^{2s}} , & x \neq y, s \in [0,1], \\ 0 & \text{otherwise} . \end{cases} \tag{4.20}$$

Fractional p-Laplacian. We are able to approximate the nonlocal fractional p-Laplacian (2.14) using the proposed operator for $\alpha = \beta = \frac{1}{2}$ as

$$\begin{aligned} \mathcal{L}_{w_3,p} f(x) &= \int_\Omega \sqrt{w_3(x,y)}^p |f(y) - f(x)|^{p-2}(f(y) - f(x))dy \\ &= \int_\Omega \frac{1}{|x-y|^{s \times p}} |f(y) - f(x)|^{p-2}(f(y) - f(x))dy . \end{aligned} \tag{4.21}$$

Remark. By using a weighting function that is nonnegative, continuous, and radial and that has the property $\sum_{u \in V}\sum_{v \in V} w(u, v) = 1$, we recover the nonlocal anisotropic p-Laplacian (2.11).

Hölder infinity Laplacian. For the case $p = \infty$ and the weighting function w_3 defined as above the proposed operator (4.1) corresponds to the recently investigated Hölder infinity Laplacian equation proposed by Chambolle, Lindgren, and Monneau in [15].

4.3. Connection with Tug-of-War game. Many local PDEs (p-Laplacian equation or infinity Laplacian equation) are related to a stochastic game called the Tug-of-War game. In the following we demonstrate that our newly introduced partial difference operator is also able to recover for $p = \infty$ the value functions of the Tug-of-War game and the biased Tug-of-War game as discussed in [46, 38, 45].

Originally, the Tug-of-War analogy was used by [46] to prove that every bounded real valued Lipschitz function F on a subset Y of a length space X (a length space is a metric space (X, d) where the distance $d(x, y)$ is the infimum of the lengths of continuous paths in X that connect x to y) admits a unique Lipschitz extension $u : X \to \mathbb{R}$ for which $Lip_U u = Lip_{\partial U} u$ for all open $U \subset X \setminus Y$. When X is the closure of a bounded domain $U \subset \mathbb{R}^n$ and Y is its boundary, a Lipschitz extension u of F is absolutely minimal if and only if it is infinity harmonic in the interior of $X \setminus Y$, i.e., it is a viscosity solution to $\Delta_\infty u = 0$ (where Δ_∞ is the infinity Laplacian).

As it has been proven that u is a Lipschitz extension of F to X by solving $\Delta_\infty u = 0$, it can be interpreted for image processing as an interpolation process. It can be used, for example, to perform semisupervised segmentation, by interpolating some initial labels on the whole image. It can also be used to perform inpainting, where the missing parts of the image can be seen as the set $X \setminus Y$. We show some illustrations of these kinds of applications in section 6.

4.3.1. Tug-of-War game. Let us briefly review the notion of the Tug-of-War game as introduced by Peres et al. [46]. Let $\Omega \subset \mathbb{R}^n$ be a Euclidean space and $g : \Omega \to \mathbb{R}$ a function. Furthermore, let $\varepsilon > 0$ be fixed. The dynamics of the game are as follows. A token is placed at an initial position $x_0 \in \Omega$. At the kth stage of the game, Player I and Player II select points x_k^I and x_k^{II}, respectively, each belonging to a specified set $B_\varepsilon(x_{k-1}) \subseteq \Omega$ (where $B_\varepsilon(x_{k-1})$ is the ε-ball centered in x_{k-1}). The game token is then moved to a new position x_k, where x_k is determined by $x_k = x_k^I$ with probability $p = \frac{1}{2}$ (otherwise, $x_k = x_k^{II}$). In other words, a fair coin is tossed to decide where the token is placed. After the kth stage of the game, if $x_k \in \Omega$, then the game continues to stage $k+1$. Otherwise, if $x_k \in \partial\Omega$, the game ends and Player II pays Player I the amount $g(x_k)$. Player I attempts to maximize the payoff while Player II attempts to minimize it. According to the dynamics programming principle, the value functions for Player I and Player II for a standard ε-turn Tug-of-War game satisfy the relation

$$f^\varepsilon(x) \;=\; \frac{1}{2}\left[\max_{y \in B_\varepsilon(x)} f^\varepsilon(y) + \min_{y \in B_\varepsilon(x)} f^\varepsilon(y)\right] \text{ on } \Omega \tag{4.22}$$

with $f^\varepsilon(x) = g(x)$ on $\partial\Omega$.

In [46] the authors show that for $\varepsilon \to 0$, $f^\varepsilon \to f$, which is a solution of the following PDE:

$$\begin{cases} \Delta_\infty^G f(x) \;=\; 0\;, & x \in \Omega, \\ f(x) \;=\; g(x)\;, & x \in \partial\Omega\;. \end{cases} \tag{4.23}$$

Using PDE (4.23), we show that the operator we propose enables us to recover the value function of this game in the graph setting. Let $G(V, E, w)$ be a Euclidean graph with $V = \Omega \subset \mathbb{R}^n$, $E = \{(x, y) \in V \times V \mid w(x, y) > 0\}$, and

$$w(x, y) = \begin{cases} 1 & \text{if } |y - x| \leqslant \varepsilon, \\ 0 & \text{otherwise}\;, \end{cases} \tag{4.24}$$

and using the following relations, easily obtained from the definition of the $\mathcal{L}^\infty$ norm of the upwind gradients in (3.12), we get

$$
\begin{aligned}
\max_{B_\varepsilon(x)} f(y) &= \|(\nabla_w^+ f)(x)\|_\infty + f(x) \;, \\
\min_{B_\varepsilon(x)} f(y) &= f(x) - \|(\nabla_w^- f)(x)\|_\infty \;.
\end{aligned}
\tag{4.25}
$$

By replacing max and min in (4.22) by their equivalent of (4.25), we get

$$
\begin{aligned}
f(x) &= \frac{1}{2}\left[\|(\nabla_w^+ f)(x)\|_\infty - \|(\nabla_w^- f)(x)\|_\infty\right] + f(x) \\
\Rightarrow\; 0 &= \frac{1}{2}\left[\|(\nabla_w^+ f)(x)\|_\infty - \|(\nabla_w^- f)(x)\|_\infty\right] \;.
\end{aligned}
\tag{4.26}
$$

For the case $\alpha = \beta = 0.5$ the proposed operator in (4.1), which is in this context the infinity Laplacian on graphs (3.23), coincides with the value function (4.22) of the Tug-of-War game:

$$
\begin{cases}
\Delta_{\infty,w} f(x) = 0 \;, & x \in \Omega, \\
f(x) = g(x) \;, & x \in \partial\Omega \;.
\end{cases}
\tag{4.27}
$$

4.4. Nonlocal Tug-of-War game. For a general Euclidean weighted graph and $\alpha = \beta = 0.5$ one can see that the proposed operator in (4.1) is connected to the following nonlocal Tug-of-War game. This is the same game as previously described, except that the ε-ball is replaced by a neighborhood $N(x_{k-1}) \subset \Omega$ defined by

$$
N(x_{k-1}) = \{x \in \Omega \mid w(x, x_{k-1}) > 0\} \cup \{x_{k-1}\}. \tag{4.28}
$$

In this nonlocal variant of the game the game token is moved to a new position x_k, where x_k is chosen arbitrarily in Ω such that $x_k = x_k^{\mathrm{I}}$ with the probability

$$
p = \frac{\sqrt{w(x_{k-1}, x_k^{\mathrm{I}})}}{\sqrt{w(x_{k-1}, x_k^{\mathrm{I}})} + \sqrt{w(x_{k-1}, x_k^{\mathrm{II}})}} \tag{4.29}
$$

and such that $x_k = x_k^{\mathrm{II}}$ with a probability $1-p$. According to the dynamic programming principle, the value functions for Player I and Player II for this game satisfy the relation

$$
\max_{y \in N(x)} \sqrt{w(x,y)}\big(f(y) - f(x)\big) + \min_{y \in N(x)} \sqrt{w(x,y)}\big(f(y) - f(x)\big) = 0, \tag{4.30}
$$

which is in our context simply

$$
\Delta_{\infty,w} f(x) = 0 \;. \tag{4.31}
$$

One can see here that the value function of the nonlocal Tug-of-War game can be found by solving this nonlocal PdE-based on the newly proposed operator.

4.5. Biased Tug-of-War. Finally, let us discuss a modified version of the Tug-of-War game as follows. Let $\alpha > 0$ and $\beta > 0$. We can add bias to the tug-of-war game by using the same game rules, but setting the probability to choose x_k^I to $p = \alpha$ and the probability for x_k^{II} as $p = \beta$. When the game is optimal, according to the dynamic principle, the corresponding value function is given by

$$\begin{cases} f^\varepsilon(x) &= \alpha \max\limits_{y \in B_\varepsilon(x)} f^\varepsilon(y) + \beta \min\limits_{y \in B_\varepsilon(x)} f^\varepsilon(y) \ , & x \in \Omega \ , \\ f(x) &= g(x) \ , & x \in \partial\Omega \ . \end{cases} \tag{4.32}$$

This value function is related to the ∞-Laplacian with gradient terms: $c|\nabla u| + \Delta_\infty u(x) = 0$, in which c depends on the values of α and β [45].

This type of PDE and the related stochastic game were studied in [45]. Now, if we consider a general Euclidean weighted graph, the probability to move the game token to x_k^I is given by

$$p = \frac{\alpha\sqrt{w(x_{k-1}, x_k^I)}}{\alpha\sqrt{w(x_{k-1}, x_k^I)} + \beta\sqrt{w(x_{k-1}, x_k^{II})}} \ , \tag{4.33}$$

and the probability for $x_k = x_k^{II}$ is $1 - p$. Following the same argumentation we get the following relation for this game:

$$\begin{cases} \mathcal{L}_{w,\infty} f(x) &= 0 \ , & x \in \Omega \ , \\ f(x) &= g(x) \ , & x \in \partial\Omega \ . \end{cases} \tag{4.34}$$

5. PdEs based on the proposed operator. Recently, many nonlocal approaches have been developed for image processing. These approaches are called nonlocal because any pixel of the image can interact directly with any other pixel in the image domain, without the usual restrictions to local interactions of a 4- or 8-neighborhood. Nonlocal models have been shown to have great advantages over many traditional local models, since local smoothness is not required for these approaches. They have also demonstrated their usefulness for geometric and repetitive structures in images (such as textures). In our previous works we have shown that the p-Laplacian-based regularization on graphs unifies both local and nonlocal diffusion filters [9]. We have also shown that the transcription of PDE-based morphology on graphs lead to nonlocal erosion and dilatation type filters [56].

In the following we study a nonlocal diffusion problem based on the proposed operator in (4.1) and we show that the $\mathcal{L}_{w,p}$ time discretization unifies p-Laplacian and morphological filtering. In particular this allows us to derive new filters that adaptatively combine both diffusion and shock filters.

We also study a family of elliptic PdEs with Dirichlet boundary conditions based on the proposed operator and prove the existence and uniqueness of respective solutions. The corresponding equation represents a generalization of interpolation processes on discrete domains.

5.1. Nonlocal diffusion equation. Given a graph $G(V, E, w)$ and a function $f : V \times [0, T] \to \mathbb{R}$, we consider the following diffusion equation for $2 \le p \le \infty$:

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} &= \mathcal{L}_{w,p} f(u,t) \ , \\ f(u, t=0) &= f_0(u) \ , \end{cases} \tag{5.1}$$

for which $f_0 : V \to \mathbb{R}$ is the initial value of f at time $t = 0$.

We discretize the derivative of f with respect to the time variable t using an explicit Euler scheme as

$$\frac{\partial f(u,t)}{\partial t} = \frac{f^{n+1}(u) - f^n(u)}{\Delta t} \tag{5.2}$$

with $f^n(u) = f(u, n\Delta t)$.

5.1.1. General case. We begin by studying the general case of $\alpha(u) \neq \beta(u) \neq 0$, $\alpha \neq 0$, provide an iterative algorithm to solve the equation, and show some properties based on the choice of p. To solve (5.1) we use the time discretization (5.2) in order to get the following general iterative scheme:

$$f^{n+1}(u) = f^n(u) + \Delta t \mathcal{L}_{w,p} f^n(u) . \tag{5.3}$$

- For the case $2 \leq p < \infty$ we get

$$f^{n+1}(u) = f^n(u) + \Delta t\big[\alpha(u)\|(\nabla_w^+ f^n)(u)\|_{p-1}^{p-1} - \beta(u)\|(\nabla_w^- f^n)(u)\|_{p-1}^{p-1}\big]. \tag{5.4}$$

Here, one can see that we recover a general nonsymmetric averaging filter which interpolates between an iterative nonlocal morphological process and an average filtering process driven by nonlocal means. We can further rewrite (5.4) as

$$\begin{aligned} f^{n+1}(u) = f^n(u) + \Delta t\Big[&\alpha(u)\sum_{v \overset{+}{\sim} u} A_{u,v,p}(f^n)(f^n(v) - f^n(u)) \\ &-\beta(u)\sum_{v \overset{-}{\sim} u} B_{u,v,p}(f^n)(f^n(u) - f^n(v))\Big] \end{aligned} \tag{5.5}$$

with $A_{u,v,p}(f) = \sqrt{w(u,v)}^{p-1}(f(v) - f(u))^{p-2}$, $B_{u,v,p}(f) = \sqrt{w(u,v)}^{p-1}(f(u) - f(v))^{p-2}$, $v \overset{+}{\sim} u = \{v \sim u | f(v) > f(u)\}$, and $v \overset{-}{\sim} u = \{v \sim u | f(v) < f(u)\}$. All the coefficients on the right-hand side are nonnegative if

$$1 \geq \Delta t(\alpha(u)\sum_{v \overset{+}{\sim} u} A_{u,v,p}(f^n) + \beta(u)\sum_{v \overset{-}{\sim} u} B_{u,v,p}(f^n)) .$$

This inequality corresponds to the well-known CFL condition for the time step Δt. This leads to maximum norm stability, and in fact to a maximum principle for this approximation to (5.1). For the whole graph we get

$$1 \geq \Delta t \max_{u \in V}\Big(\alpha(u)\sum_{v \overset{+}{\sim} u} A_{u,v,p}(f^n) + \beta(u)\sum_{v \overset{-}{\sim} u} B_{u,v,p}(f^n)\Big) . \tag{5.6}$$

Therefore, considering the upper bound of the right-hand term of (5.6), we get

$$1 \geq \Delta t \max_{u \in V}(|N(u)| \max_{v \sim u}(|f(v) - f(u)|^{p-2}) .$$

Based on the previous inequality we can determine the maximum for Δt as

$$\Delta t^* \;=\; \frac{1}{\max\limits_{u\in V}(|N(u)|\max\limits_{v\sim u}(|f(v)-f(u)|^{p-2}))} \;. \tag{5.7}$$

In order to make the proposed operator more interpretable, we introduce the two following additional operators:

$$\begin{aligned} NLD_p(f)(u) \;&=\; f(u) \;+\; \tau\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} \;, \\ NLE_p(f)(u) \;&=\; f(u) \;-\; \tau\|(\nabla_w^- f)(u)\|_{p-1}^{p-1} \;, \end{aligned} \tag{5.8}$$

for which $\tau = \Delta t^*$ and $NLD_p, NLE_p : \mathcal{H}(V) \to \mathcal{H}(V)$ represent nonlocal dilation and nonlocal erosion, respectively. Since we have the identity $\|(\nabla_w^+ f)(u)\| = \|(\nabla_w^- - f)(u)\|$ these two operators have the following useful property:

$$\begin{aligned} NLD_p(-f) \;&=\; -NLE_p(f) \;, \\ NLE_p(-f) \;&=\; -NLD_p(f) \;. \end{aligned} \tag{5.9}$$

Now we can set an iteration step of the scheme (5.5) as

$$f^{n+1}(u) = NLA_p(f^n)(u), \tag{5.10}$$

where NLA denotes nonlocal averaging defined by $NLA_p(f)(u) = \alpha(u)NLD_p(f)(u) + \beta(u)NLE_p(f)(u)$. If we use these formulation we finally get

$$\begin{aligned} NLA_p(f)(u) \;&=\alpha(u)NLD_p(f)(u) \;+\; \beta(u)NLE_p(f)(u) \\ &= \quad \alpha(u)\big(f(u) + \tau\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1}\big) \\ &\quad + \;\; \beta(u)(f(u) - \tau\|(\nabla_w^- f)(u)\|_{p-1}^{p-1}) \\ &= \quad (\alpha(u) + \beta(u))f(u) \\ &\quad + \tau\alpha(u)\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} \;-\; \tau\beta(u)\|(\nabla_w^- f)(u)\|_{p-1}^{p-1} \\ &= f(u) + \tau\big[\alpha(u)\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} \;-\; \beta(u)\|(\nabla_w^- f)(u)\|_{p-1}^{p-1}\big] \;, \end{aligned} \tag{5.11}$$

which corresponds to our iterative scheme (5.3).

- For the case $p = \infty$ we can write

$$f^{n+1}(u) \;=\; f^n(u) + \Delta t\big[\alpha(u)\|(\nabla_w^+ f^n)(u)\|_\infty \;-\; \beta(u)\|(\nabla_w^- f^n)(u)\|_\infty\big] \;. \tag{5.12}$$

As in the case $2 \le p < \infty$ discussed above, we get the following condition at a node u to ensure stability of the scheme:

$$1 \;\ge\; \Delta t(\alpha(u)\sqrt{w(u,v_0)} \;+\; \beta(u)\sqrt{w(u,v_1)}$$

with

$$\begin{aligned} v_0 \;&=\; \arg\max_{v \overset{+}{\sim} u}(\sqrt{w(u,v)}f(v) - f(u)) \qquad \text{and} \\ v_1 \;&=\; \arg\max_{v \overset{-}{\sim} u}(\sqrt{w(u,v)}f(u) - f(v)) \;. \end{aligned}$$

Since the maximum value of the weighting function w is 1, and $\alpha(u) + \beta(u) = 1$, we simply get $\Delta t \leq 1$. Considering the iterative scheme (5.12) to be stable for $\Delta t = 1$, we define the two operators $NLD_\infty : \mathcal{H}(V) \to \mathcal{H}(V)$ and $NLE_\infty : \mathcal{H}(V) \to \mathcal{H}(V)$ as

$$\begin{aligned} NLD_\infty(f)(u) &= f(u) + \|(\nabla_w^+ f)(u)\|_\infty , \\ NLE_\infty(f)(u) &= f(u) - \|(\nabla_w^- f)(u)\|_\infty . \end{aligned} \tag{5.13}$$

Now we can rewrite an iteration of (5.12) as

$$f^{n+1}(u) = NLA_\infty(f^n)(u) \tag{5.14}$$

with $NLA_\infty : \mathcal{H}(V) \to \mathcal{H}(V)$ defined as

$$\begin{aligned} NLA_\infty(f)(u) &= \alpha(u) NLD_\infty(f)(u) + \beta(u) NLE_\infty(f)(u) \\ &= \alpha(u)(f(u) + \|(\nabla_w^+ f)(u)\|_\infty) + \beta(u)(f(u) - \|(\nabla_w^- f)(u)\|_\infty) \\ &= f(u) + \alpha(u)\|(\nabla_w^+ f)(u)\|_\infty - \beta(u)\|(\nabla_w^- f)(u)\|_\infty , \end{aligned} \tag{5.15}$$

which corresponds to our iterative scheme (5.12).

5.1.2. Special cases of filters. In this section we show that special cases of the iteration scheme (5.4) permit us to recover nonlocal image processing filters we have introduced in our previous works. Indeed, using the proposed p-Laplacian operator with gradient terms, we can provide an interpretation of morphological operators as a family of nonlocal digital averaging filters that can be expressed using the previously described iterative scheme.

- For the cases $\alpha(u) = 0$ or $\beta(u) = 0$ we recover filters related to PdE-based morphology:

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} = \pm\|(\nabla_w^\pm f)(u,t)\|_{p-1}^{p-1} & \text{for } 2 \leq p < \infty , \\ \frac{\partial f(u,t)}{\partial t} = \pm\|(\nabla_w^\pm f)(u,t)\|_\infty & \text{for } p = \infty . \end{cases} \tag{5.16}$$

For $\beta(u) = 0$ we get the following PdEs, which correspond to *nonlocal discrete dilation* on graphs:

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} = \|(\nabla_w^+ f)(u,t)\|_{p-1}^{p-1} & \text{for } 2 \leq p < \infty , \\ \frac{\partial f(u,t)}{\partial t} = \|(\nabla_w^+ f)(u,t)\|_\infty & \text{for } p = \infty . \end{cases} \tag{5.17}$$

Expressed by the previously introduced operators NLD_p we get the following iterative scheme:

$$\begin{cases} f^{n+1}(u) = NLD_p(f^n)(u) & \text{for } 2 \leq p < \infty , \\ f^{n+1}(u) = NLD_\infty(f^n)(u) & \text{for } p = \infty . \end{cases} \tag{5.18}$$

For $\alpha(u) = 0$ we get PdEs corresponding to nonlocal discrete erosion on graphs:

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} = -\|(\nabla_w^- f)(u,t)\|_{p-1}^{p-1} & \text{for } 2 \leq p < \infty , \\ \frac{\partial f(u,t)}{\partial t} = -\|(\nabla_w^- f)(u,t)\|_\infty & \text{for } p = \infty . \end{cases} \tag{5.19}$$

Expressed by the previously introduced operators NLE_p we get the following iterative scheme:

$$\begin{cases} f^{n+1}(u) &= NLE_p(f^n)(u) & \text{for } 2 \le p < \infty \ , \\ f^{n+1}(u) &= NLE_\infty(f^n)(u) & \text{for } p = \infty \ . \end{cases} \tag{5.20}$$

Note that for $w(u,v) = 1 \forall (u,v) \in E$, and $p = \infty$, we recover the traditional discrete morphological operators [11]—erosion for $\alpha = 0$ and dilation for $\beta = 0$.

- For the special case $\alpha(u) = \beta(u)$ we can express (5.1) as

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} &= \Delta^a_{w,p} f(u) & \text{for } 2 \le p < \infty \ , \\ \frac{\partial f(u,t)}{\partial t} &= \Delta_{w,\infty} f(u) & \text{for } p = \infty \ . \end{cases} \tag{5.21}$$

Using the iterative scheme (5.10) for $2 \le p < \infty$ we get

$$f^{n+1}(u) \ = \ NLM_p(f^n)(u) \ , \tag{5.22}$$

for which the operator $NLM_p : \mathcal{H}(V) \to \mathcal{H}(V)$ is defined as

$$\begin{aligned} NLM_p(f)(u) &= \frac{1}{2}(NLD_p(f)(u) + NLE_p(f)) \\ &= f(u) + \frac{\tau}{2}(\|(\nabla^+_w f)(u)\|^{p-1}_{p-1} \ + \ \|(\nabla^-_w f)(u)\|^{p-1}_{p-1}) \\ &= f(u) \ + \ \tau \Delta^a_{w,p} f(u) \ , \end{aligned} \tag{5.23}$$

and the maximum time step width $\tau = \Delta t^*$ from (5.7).
For the case $p = \infty$ we derive the following iterative scheme:

$$f^{n+1}(u) \ = \ NLM_\infty(f^n)(u) \ , \tag{5.24}$$

where the operator $NLM_\infty : \mathcal{H}(V) \to \mathcal{H}(V)$ is defined as

$$\begin{aligned} NLM_\infty(f)(u) &= \frac{1}{2}(NLD_\infty(f)(u) \ + \ NLE_\infty(f)) \\ &= f(u) + \frac{1}{2}(\|(\nabla^+_w f)(u)\|_\infty - \|(\nabla^-_w f)(u)\|_\infty) \\ &= f(u) \ + \ \Delta_{w,\infty} f(u) \ . \end{aligned} \tag{5.25}$$

5.1.3. Properties. In the following we give important properties of the filtering process (5.5) introduced above by rewriting the respective operators to matrices. Subsequently, we show that the iterative scheme satisfies the minimum-maximum principle (MMP). Finally, we demonstrate that convergence of the related diffusion process to a function f^* leads to $\mathcal{L}_{w,p} f^*(u) = 0$.

We begin by stating that for a given function $f : V \to \mathbb{R}$ the iteration process (5.10) can be written as

$$F^{n+1} \ = \ \Phi(F^n) F^n, \tag{5.26}$$

where $F^n = \big(f^n(u)\big)^T_{u\in V} \in \mathbb{R}^N$ and $\Phi(F^n)$ is the matrix consisting of all coefficients with respect to the weighting function, the parameters α and β, and the vector F at iteration n. It can be explicitly given by

$$\Phi(F^n)(u,v) = \begin{cases} 1 - \tau\big(\alpha(u) \sum\limits_{z \overset{+}{\sim} u} A_{u,z,p}(f^n) + \beta(u) \sum\limits_{z \overset{-}{\sim} u} B_{u,z,p}(f^n)\big) & \text{if } u = v, \\ \tau\big(\alpha(u) A_{u,v,p}(f^n) + \beta(u) B_{u,v,p}(f^n)\big) & \text{if } v \sim u, \\ 0 & \text{otherwise.} \end{cases} \tag{5.27}$$

It is easy to show that the matrix Φ has the following properties:

- The sum of every line of Φ is equal to 1.
- The matrix is nonnegative: $\Phi(F^n)(u,v) \geq 0 \forall (u,v) \in E$.
- Unless $\alpha(u) = \beta(u)\ \forall\ u$ in V, the average value of f is not preserved.

Proposition 5.1. *The iterative filtering scheme* (5.10) *satisfies the MMP.*

Proof. Let $m = \min_{u\in V}\big(f^0(u)\big)$ and $M = \max_{u\in V}\big(f^0(u)\big)$. By definition we know that the nonlocal operators NLE_p and NLD_p (see (5.8)) satisfy $\forall\ u \in V$

$$\begin{aligned} m \ &\leq \ NLE_p\big(f^0\big)(u) \ \leq \ M\ , \\ m \ &\leq \ NLD_p\big(f^0\big)(u) \ \leq \ M\ . \end{aligned} \tag{5.28}$$

According to these inequalities, and recalling that the parameters $\alpha(u) + \beta(u) = 1$, we can write $\forall\ u \in V$

$$m \ \leq \ \alpha(u) NLE_p\big(f^0\big)(u) \ + \ \beta(u) NLD_p\big(f^0\big)(u) \ \leq \ M \tag{5.29}$$

and thus

$$m \ \leq \ NLA\big(f^0(u)\big) \ \leq \ M\ . \tag{5.30}$$

Finally, by induction this relation can be extended to any subsequent time step n. ■

We can conclude that the scheme (5.10) is stable and corresponds to a nonlocal filtering process that combines dilation, erosion, and nonlocal mean. In case of a graph $G = (V, E, w)$ composed of N vertices and a function $f_0 \in \mathcal{H}(V)$, a simple filtering process can then be written using the following algorithm:

1. Vertices are ordered linearly. We have $u_1 < u_2 < \cdots < u_N$.
2. The algorithm is initialized with $f^0 = f_0$.
3. For every $k = 1, \ldots, N$ do $f^{n+1}(u_k) = NLA_p(f^n(u_k))$.

Proposition 5.2. *If the iterative filtering process* (5.10) *converges to a function f^*, then f^* satisfies $\mathcal{L}_{w,p} f^*(u) = 0\ \forall u \in V$.*

Proof. Let f^* be the limit of the iterative scheme (5.10). Then we have

$$\begin{aligned} f^*(u) \ &= \ NLA(f^*)(u) \\ &= \ f^*(u) + \tau[\alpha(u) \|(\nabla^+_w f^*)(u)\|^{p-1}_{p-1} - \beta(u) \|(\nabla^-_w f^*)(u)\|^{p-1}_{p-1}] \\ \Rightarrow \quad 0 \ &= \ \tau \mathcal{L}_{w,p} f^*(u). \end{aligned} \tag{5.31}$$

As the graphs we use are composed of a finite set of edges and $\tau > 0$ we can already deduce that $\mathcal{L}_{w,p} f^*(u) = 0$. ■

5.2. Dirichlet problem. In the following we focus on the Dirichlet problem associated to the p-Laplacian with gradient terms $\mathcal{L}_{w,p}f$ and show that this problem has a unique solution. Broadly speaking, the Dirichlet problem is a boundary value problem of the following type: find a function $f \in \mathcal{H}(V)$ such that $\mathcal{L}_{w,p}f = 0$ on a set $A \subset V$, knowing the value of f on the boundary ∂A.

Let $G = (V, E, w)$ a weighted and connected graph, $A \subset V$ a set of vertices, and $g : \partial A \to \mathbb{R}$ a function defined on the boundary of A. We consider the following equation that describes the Dirichlet problem associated to our newly introduced nonlocal Laplacian operator:

$$\begin{cases} \mathcal{L}_{w,p}f(u) \;=\; 0 & \text{for } u \in A\ , \\ f(u) \;=\; g(u) & \text{for } u \in \partial A\ . \end{cases} \tag{5.32}$$

Many problems in image processing and machine learning can be formulated as this kind of interpolation problem. In this part we will only study the case $2 \leq p < \infty$ as the case $p = \infty$ has already been studied in our work in [21].

5.2.1. Proof of existence and uniqueness.

Theorem 5.3. *Given a graph $G = (V, E, w)$, a set $A \subset V$, and a function $g : \partial A \to \mathbb{R}$, there exists a unique function $f \in \mathcal{H}(V)$ such that f verifies the following equation:*

$$\begin{cases} \alpha(u)\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} - \beta(u)\|(\nabla_w^- f)(u)\|_{p-1}^{p-1} \;=\; 0 & \textit{for } u \in A\ , \\ f(u) \;=\; g(u) & \textit{for } u \in \partial A\ . \end{cases} \tag{5.33}$$

Proof. First, we note that (5.33) can be rewritten as $f(u) = \mathcal{L}_{w,p}f(u)$. We will begin by proving the uniqueness of respective solutions by using the comparison principle. Given two functions f and h, we will prove that if $f = \mathcal{L}_{w,p}f$ and $h = \mathcal{L}_{w,p}h$ with $f \leq h$ on ∂A, then, $f \leq h$ on the whole domain V. In order to deduce a contradiction, we assume that there exists an $M \in \mathbb{R}$ such that

$$M \;=\; \sup_V (f - h) \;>\; 0\ .$$

Let $B = \{u \in A : f(u) - h(u) = M\}$. By construction we have $B \neq \emptyset$ and $B \cap \partial A = \emptyset$. We claim that there exists $u_0 \in B$ and $v \in N(u_0)$, such that $v \notin B$. Otherwise, if for each $u \in A$ and for each $v \in N(u)$ we have $v \notin B$, then this implies that $B \cap \partial A \neq \emptyset$, since the graph is connected and thus we have a contradiction.

Then, from the definition of M we have

$$\begin{aligned} f(u_0) - h(u_0) \geq f(u) - h(u) \qquad & \forall u \in N(u_0)\ , \\ h(u) - h(u_0) \geq f(u) - f(u_0) \qquad & \forall u \in N(u_0)\ . \end{aligned}$$

In particular we can write

$$h(v) - h(u_0) \;>\; f(v) - f(u_0)\ .$$

From these inequalities, we can deduce

$$
\begin{aligned}
\max(h(u)-h(u_0),0) \;&\geq\; \max(f(u)-f(u_0),0),\\
(\sqrt{w(u_0,u)}\max(h(u)-h(u_0),0))^p &\geq (\sqrt{w(u_0,u)}\max(f(u)-f(u_0),0))^p,\\
\alpha(u_0)\sum_{u\sim u_0}(\sqrt{w(u_0,u)}\max(h(u)-h(u_0),0))^p &> \alpha(u_0)\sum_{u\sim u_0}(\sqrt{w(u_0,u)}\max(f(u)-f(u_0),0))^p,\\
\alpha(u_0)\|(\nabla_w^+h)(u_0)\|_p^p &> \alpha(u_0)\|(\nabla_w^+f)(u_0)\|_p^p,
\end{aligned}
\tag{5.34}
$$

and analogously

$$
\begin{aligned}
h(u_0)-h(u) &\leq f(u_0)-f(u),\\
\max(h(u)-h(u_0),0) &\geq \max(f(u)-f(u_0),0),\\
\sum_{u\sim u_0}(\sqrt{w(u_0,u)}\max(h(u_0)-h(u),0))^p &< \sum_{u\sim u_0}(\sqrt{w(u_0,u)}\max(f(u_0)-f(u),0))^p,\\
\|(\nabla_w^-h)(u_0)\|_p^p &< \|(\nabla_w^-f)(u_0)\|_p^p,\\
-\beta(u_0)\|(\nabla_w^-h)(u_0)\|_p^p &> -\beta(u_0)\|(\nabla_w^-f)(u_0)\|_p^p.
\end{aligned}
\tag{5.35}
$$

The previous inequalities are strict inequalities because we know there exists a $v \in N(u_0)$ such that $h(v)-h(u_0) > f(v)-f(u_0)$. Using the relations (5.34) and (5.35) we can deduce the following inequality:

$$
\begin{aligned}
\alpha(u_0)\|(\nabla_w^+h)(u_0)\|_p^p-\beta(u_0)\|(\nabla_w^-h)(u_0)\|_p^p &> \alpha(u_0)\|(\nabla_w^+f)(u_0)\|_p^p-\beta(u_0)\|(\nabla_w^-f)(u_0)\|_p^p,\\
\mathcal{L}_{w,p}h(u_0) &> \mathcal{L}_{w,p}f(u_0),\\
0 &> 0.
\end{aligned}
$$

This clearly leads to a contradiction and concludes the proof of uniqueness.

For the proof of existence of respective solutions we recall the Brouwer fixed point theorem. It states that a continuous function defined on a convex, compact subset of a Euclidean space which maps into the same subset has a fixed point. We identify $\mathcal{H}(V)$ as $\mathbb{R}^n$ and consider the set $K = \{f \in \mathcal{H}(V) \mid f(u) = g(u)\ \forall u \in \partial A, \text{ and } m \leqslant f(u) \leqslant M\ \forall u \in A\}$, where $m = \min_{\partial A}\big(g(u)\big)$ and $M = \max_{\partial A}\big(g(u)\big)$. By definition, K is a convex and a compact subset of $\mathbb{R}^n$. Now, it is easy to show that the map $f \to NLA_p\big(f\big)$ is continuous and maps from K to K. Using the Brouwer fixed point theorem we can deduce that the map NLA_p has a fixed point, which is the solution of $NLA_p\big(f\big) = f$. This completes the proof. ∎

6. Applications to inverse problems on weighted graphs. In the following we apply the newly proposed nonlocal p-Laplacian with gradient terms on different inverse problems such as function restoration or interpolation on graphs. Note that in this section it is not our aim to compare our approach to state-of-the-art methods with respect to its performance in particular applications but rather to illustrate the potential of this universal formulation. In particular, we will investigate the impact of different parameter choices for α and β.

6.1. Graph construction. There exist several popular methods to transform discrete data $\{x_1,\dots,x_n\}$ into a weighted graph structure. Considering a set of vertices V such that the data are embedded by functions of $\mathcal{H}(V)$, the construction of such a graph consists in modeling the neighborhood relationships between the data through the definition of a set of edges E and using a pairwise distance measure $\mu : V \times V \to \mathbb{R}^+$. In the particular case of images, graph construction methods based on geometric neighborhoods are particularly well-adapted to represent the geometry of the space, as well as the geometry of the function defined on that space. We distinguish the following types of graphs:

- *Grid graphs*, which are the most natural structures to describe an image with a graph. Each pixel is connected by an edge to its adjacent pixels. Classical grid graphs are 4-adjacency grid graphs and 8-adjacency grid graphs. Larger adjacency can be used to obtain nonlocal grid graphs.
- *Region adjacency graphs* (RAGs), which provide very useful ways of describing the structure of a picture: vertices represent regions and edges represent region adjacency relationship.
- *k-nearest neighborhood graphs* (k-NNGs), where each vertex u is connected with its k-nearest neighbors according to the distance measure μ. Such construction implies building a directed graph as the neighborhood relationship is not symmetric. Nevertheless, an undirected graph can be obtained by adding an edge between two vertices u and v if u is among the k-nearest neighbors of v or if v is among the k-nearest neighbors of u.
- *k-extended RAGs* (k-ERAGs), which are RAGs extended by a k-NNG. Each vertex is connected to adjacent regions vertices and to its k most similar vertices of V.

The similarity between two vertices is computed with respect to an appropriate measure $s : E \to \mathbb{R}^+$, which satisfies

$$w(u, v) \;=\; \begin{cases} s(u, v) & \text{if } (u, v) \in E \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Examples for common similarity functions are as follows:

$$\begin{aligned} s_0(u, v) &= 1 \ , \\ s_1(u, v) &= \exp\Big(-\mu\big(f^0(u), f^0(v)\big)/\sigma^2\Big) \ \text{ with } \ \sigma > 0 \ , \end{aligned}$$

for which σ depends on the variation of the function μ and controls the similarity scale.

Several choices can be considered as feature vectors computed from the given data, depending on the nature of the features to be used for graph processing. In the context of image processing one can use the simple grayscale or color feature vector F_u, or a patch feature vector $F_u^\tau = \bigcup_{v \in \mathcal{W}^\tau(u)} F_v$ (i.e., the set of values F_v, where v is in a square window $\mathcal{W}^\tau(u)$ of size $(2\tau + 1) \times (2\tau + 1)$ centered at a vertex pixel u) incorporating nonlocal features such as texture.

6.2. Image restoration and simplification. We begin by demonstrating the diffusion process based on the p-Laplacian with gradient terms for filtering real image data, defined both on two-dimensional regular grids and three-dimensional (3D) point clouds.

An image of N pixels can be interpreted as a discrete function $f_0 : V \to \mathbb{R}^M$, which defines a mapping from the vertices to the color space of dimension M. Figure 1 shows exemplary results obtained for different graph constructions (local and nonlocal), built from an original noisy image, and different values of the parameters α and β. The first column shows the results obtained on an 8-adjacency grid graph for $w = s_0$, while the second column uses $w = s_1$. In both cases μ is chosen as the Euclidean distance in the color space of the image. The third column shows results obtained for a nonlocal graph, using a 15×15 neighborhood window and 5×5 patches as features vector (the weight function holds similarily between patches, with $w = s_1$ and μ is the Euclidean distance between patches). Note that the first and third rows show results for $\alpha = \beta = 0.5$, which corresponds to the anisotropic p-Laplacian diffusion process (5.22) and (5.24) (for the cases $p = 2$ and $p = \infty$, respectively).

Adaptive p-Laplacian diffusion and morphological smoothing. In the following we propose to take advantage of the α and β functions to obtain an optimal trade-off between the diffusion part and morphological smoothing, depending on the mean curvature of the graph. For this, we recall the definition of the mean curvature on graphs as introduced in [14]:

$$\kappa_w(u, f) \;=\; \frac{\sum_{v\sim u} \sqrt{w(u,v)}\operatorname{sign}(f(v) - f(u))}{\sum_{v\sim u} \sqrt{w(u,v)}} \;, \tag{6.1}$$

where the sign function is defined as

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \;, \\ -1 & \text{if } x < 0 \;, \end{cases} \tag{6.2}$$

which can be rewritten as

$$\kappa_w(u, f) \;=\; \frac{\sum_{v\in\{v|f(v)\geq f(u)\}} \sqrt{w(u,v)} - \sum_{v\in\{v|f(v)< f(u)\}} \sqrt{w(u,v)}}{\sum_{v\sim u} \sqrt{w(u,v)}} \;. \tag{6.3}$$

In order to fulfill the constraint $\alpha(u) + \beta(u) = 1$, we adapted this formulation so that the α values represent the positive part of the curvature and the β values the negative part, which leads to

$$\alpha(u) \;=\; \frac{\sum_{v\in\{v|f(v)\geq f(u)\}} \sqrt{w(u,v)}}{\sum_{v\sim u} \sqrt{w(u,v)}} \tag{6.4}$$

and

$$\beta(u) \;=\; \frac{\sum_{v\in\{v|f(v)< f(u)\}} \sqrt{w(u,v)}}{\sum_{v\sim u} \sqrt{w(u,v)}} \;. \tag{6.5}$$

Let us rewrite the diffusion process (5.1) by using (4.2) and (6.1):

$$\begin{cases} \frac{\partial f(u,t)}{\partial t} = & 2\min(\alpha(u), \beta(u))\Delta_{w,p} f(u) \\ & + \; (\kappa_w(u,f))^{+} \|(\nabla_w^{+} f)(u)\|_{p-1}^{p-1} \\ & - \; (\kappa_w(u,f))^{-} \|(\nabla_w^{-} f)(u)\|_{p-1}^{p-1}, \\ f(u, t=0) = & f_0(u) \end{cases} \tag{6.6}$$

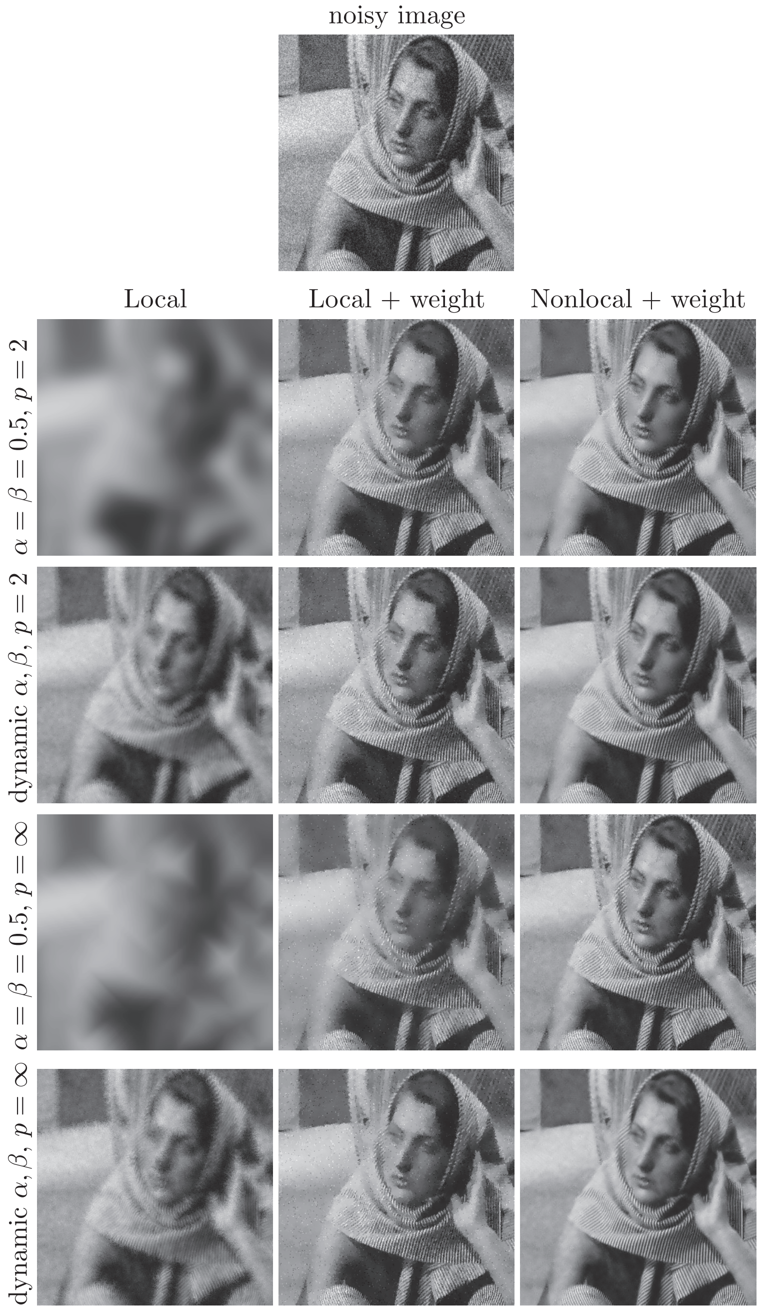

Figure 1. *Illustration of the effect of the proposed p-Laplacian with gradient terms for image filtering. The three columns represent different graph constructions, i.e., a four-grid graph with $w = s_0$ for the first column, a four-grid graph with $w = s_1$ using color similarity for the second column, and a k-nn graph using patch similarity for the third column. The first and third rows show the effect of using $\alpha = \beta = 0.5$, which corresponds to the anisotropic p-Laplacian on graphs. The second and fourth rows show adaptive α and β values based on the mean curvature of the graphs.*

For $\kappa_w(u,f) > 0$ we get

$$\frac{\partial f(u,t)}{\partial t} = 2\beta(u)\Delta_{w,p}f(u) + \kappa_w(u,f)\|(\nabla_w^+ f)(u)\|_{p-1}^{p-1} . \tag{6.7}$$

For $\kappa_w(u,f) < 0$ we have

$$\frac{\partial f(u,t)}{\partial t} = 2\alpha(u)\Delta_{w,p}f(u) + \kappa_w(u,f)\|(\nabla_w^- f)(u)\|_{p-1}^{p-1} . \tag{6.8}$$

And finally for $\kappa_w(u,f) = 0$ we get

$$\frac{\partial f(u,t)}{\partial t} = \Delta_{w,p}f(u) . \tag{6.9}$$

This enables us to adjust the filtering process adaptively between erosion and dilation, based on the data mean curvature, which allows us to combine smoothing (with the Laplacian term) and shock filtering (with either dilation or erosion) in the same formulation. Results using the dynamic adaptation of α and β for image filtering are shown in the second and fourth rows of Figure 1 for $p = 2$ and $p = \infty$, respectively.

Another illustration is given in Figure 2, which shows several results obtained on a 3D point cloud with different values of the parameters α and β. This application uses a non-Euclidean graph built as a $k-$nn graph from the set $\Omega \subset \mathbb{R}^3$ of points of a given 3D point cloud. The first and second columns show the results obtained from a spatial k-nn graph construction, with $w = s_0$ for the first column and $w = s_1$ for the second one. In both cases μ is the Euclidean distance between the color information of each point. In the third column we built a graph using the patches on 3D point cloud data as proposed in our work in [37]. The first row shows results for $\alpha = \beta = 0.5$ with $p = 2$, which corresponds to a nonlocal diffusion process (5.22). The second row shows results with α and β values depending on the graph curvature, with $p = 2$, which corresponds to a combination of smoothing and shock filtering, described in (6.6).

Figure 3 shows the morphological processes we can recover using either $\alpha = 0$ in the first and third rows (corresponding to the erosion process described in (5.20)) or $\beta = 0$ in the second and fourth rows (corresponding to the dilation process described in (5.18)). On the first two rows we show the effect of using the proposed operator on a grid graph with $w(u,v) = 1$. One can see that we recover the effects of classical erosion and dilation. In the third and fourth rows we use a weighted grid graph with $w = s_1$. As one can see this enables us to perform erosion and dilation while preserving certain details in the image.

6.3. Interpolation. Many tasks in image processing, computer vision, and machine learning can be formulated as interpolation problems. Image and video colorization, inpainting, and semisupervised segmentation/clustering are examples of these interpolation problems. Interpolating data consists in constructing new values for missing data in coherence with a set of known data. In this paper we propose to use the new p-Laplacian with gradient terms as a unified framework for the solution for both semisupervised segmentation/clustering and image inpainting. In this context we solve the following Dirichlet problem:

$$\begin{cases} \mathcal{L}_{w,p}(f)(u) = 0 & \text{for } u \in V_0 , \\ f(u) = g(u), & u \in V - V_0 = \partial V_0, \end{cases} \tag{6.10}$$

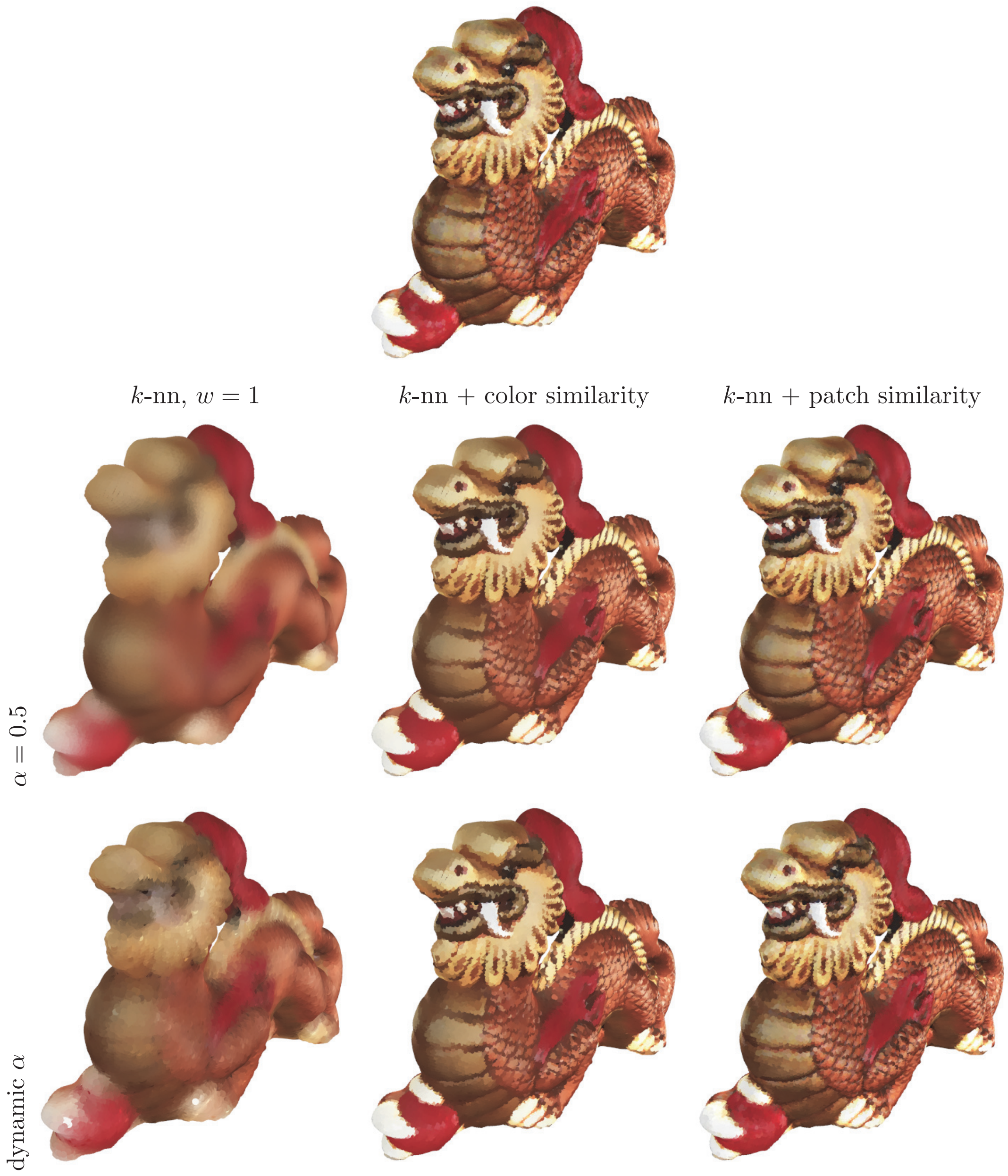

Figure 2. *Colored 3D point cloud simplification for different weighting functions and $p = 2$. The graph is built as a k-nn graph in the 3D coordinates space. The second row shows the results using $\alpha = \beta = 0.5$, which corresponds to anisotropic regularization based on the anisotropic 2-Laplacian. In the third row we show results using α and β depending on the mean curvature of the graph.*

where $V_0 \subset V$ is the subset of vertices representing the missing information. The initial value function g is application-dependent and will be discussed in more detail for each application in the following.

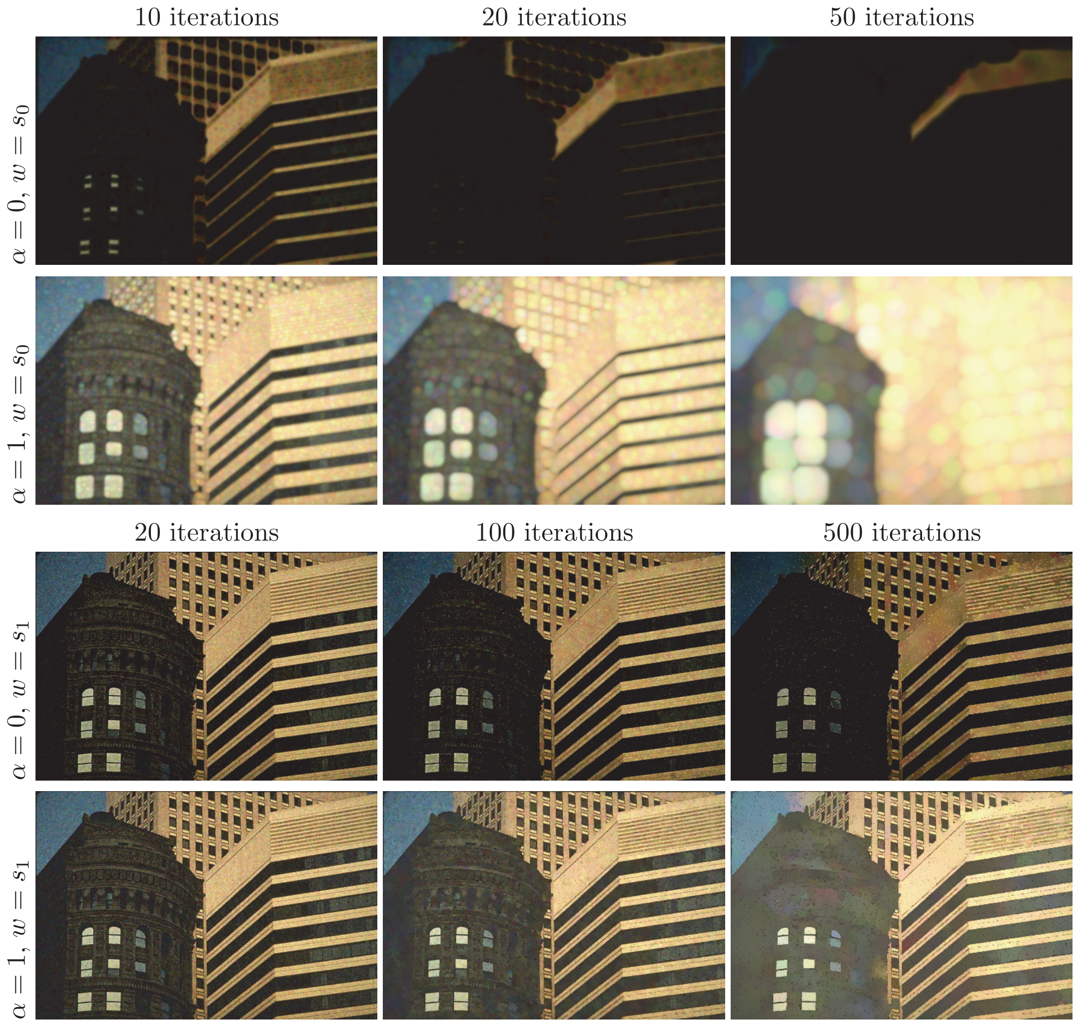

Figure 3. *Illustration of different numbers of iterations during morphological processes using different values of α and β for different local graph constructions and $p = 2$. See text for more details.*

6.3.1. Active contours, semisupervised segmentation, and classification. In the case of semisupervised image segmentation, graph-based approaches have became very popular in recent years. Many graph-based algorithms for image segmentation have been proposed, such as graph-cuts [10], random walker [29], shortest-paths [5, 25], watershed [8, 17, 58], or frameworks, that unify some of the previous methods (such as power-watershed) [16, 52]. Recently, these algorithms were all placed into a common framework [16] that allows them to be seen as special cases of a single general semisupervised algorithm. Several popular approaches [5, 17, 18, 25, 39] perform graph clustering by computing a graph partition from the set of user's seeds and a metric. We refer interested readers to [18] for more details.

In this paper we propose to consider this particular problem in two different ways. First, we deal with an interpolation problem, where the function to interpolate is the label function.

Using (6.10) and considering two classes A and B, the initial value label function g is defined as follows:

$$\begin{cases} g(u) = -1 & \text{if } u \in A\ , \\ g(u) = 1 & \text{if } u \in B\ , \\ g(u) = 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

At convergence the class membership can be easily computed by a simple threshold on the sign of f.

Remark. In the case of more than two classes, multiclasses segmentation can be performed by several segmentations of one class versus the others.

Second, we consider segmentation by curve evolution processes in which the segmentation is implicitly represented by the zero level set of an appropriate function, partitioning the graph into two sets. To adapt this idea to our formulation we first define the initial level set function ϕ_0 for two vertex sets A and B with $A \cup B = V$ as

$$\begin{cases} \phi_0(u) = -1 & \text{if } u \in A\ , \\ \phi_0(u) = 1 & \text{if } u \in B\ . \end{cases} \tag{6.12}$$

The curve evolution process can now be given as

$$\begin{cases} \frac{\partial \phi(u,t)}{\partial t} = \mathcal{L}_{w,p}(\phi)(u)\ , \\ \phi(u,0) = \phi_0(u)\ . \end{cases} \tag{6.13}$$

Active contours. To formulate an active contours algorithm on graphs to perform curve evolution we used (6.13) and adaptively set the α and β values as the mean curvature of the graph. Figure 4 presents a result of the active contour method based on the mean curvature. The graph is built as an augmented adjacency graph, i.e., we first compute a grid graph, then we add a certain amount of randomly chosen neighbors in the window around the considered pixel. The weight function depends on the similarity between pixel color information. This allows us to get a nonlocal graph without adding too many edges while keeping its size relatively small.

Interactive image segmentation. In the following we investigate label interpolation for semisupervised image segmentation. In this case the graph is the same graph as for the active contour method discussed above, and the weighting function depends again on the similarity between pixel colors. The function $f_0 : V \to \mathbb{R}$ to be interpolated is initialized according to user-defined seeds, as presented in (6.11). Figure 5 presents the initial image with user-defined seeds (blue and green) and the result of interpolation for $p = 2$, with $\alpha = \beta$, recovering the anisotropic Laplacian $\Delta_{w,2}$ (4.3), and $w = s_1$.

RAG segmentation. In the following we discuss label interpolation for semisupervised image segmentation using RAGs. This illustrates the adaptivity of the proposed approach to process irregular graphs and an efficient way to extract similar but not connected objects with only a few seeds. In this case, the graph is a RAG built from the initial image using the super vertices approach presented in [18], which is extended with a k-nn graph in order to add supplementary edges between each region and its k most similar regions in the whole

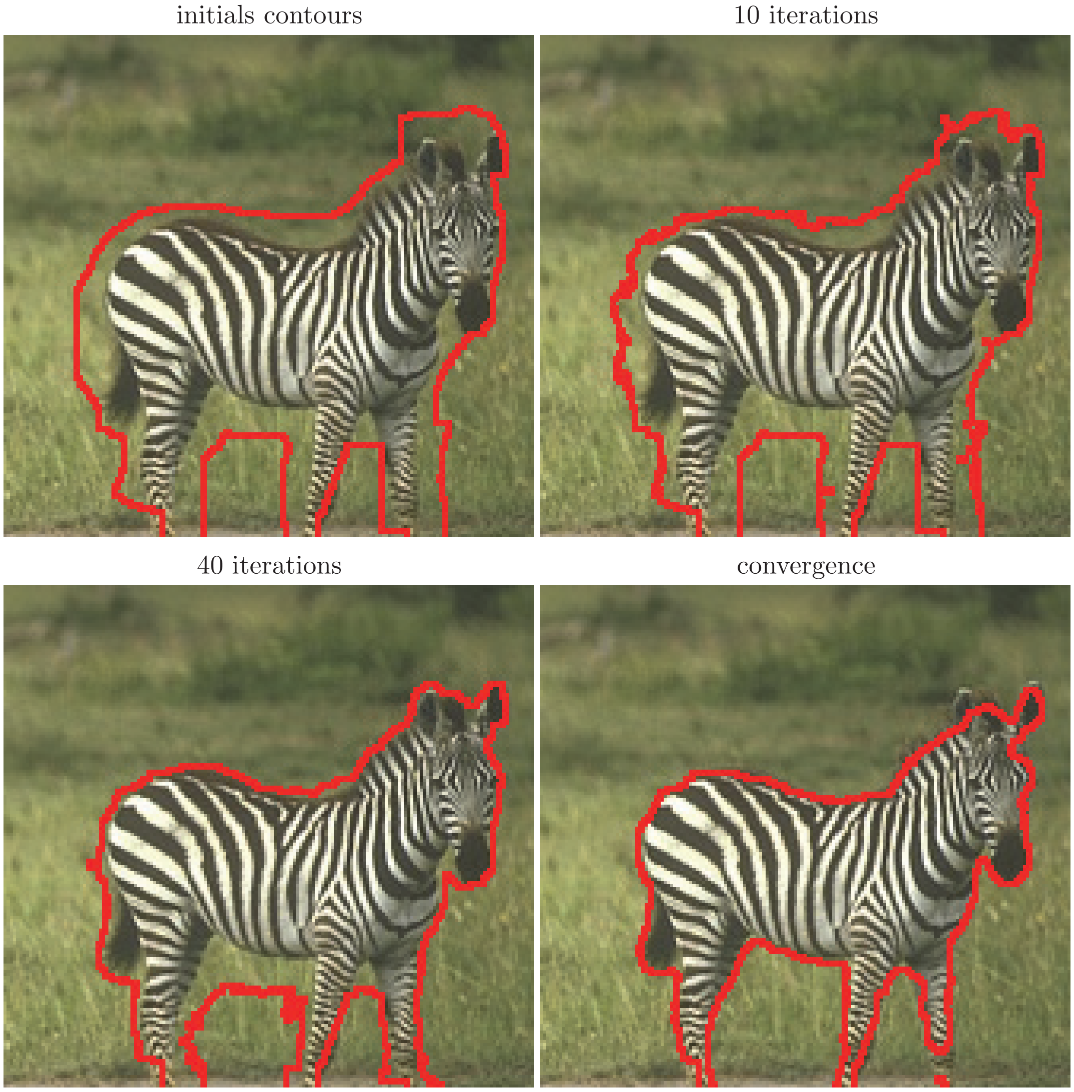

Figure 4. *Illustration of different iterations of the active contour algorithm based on mean curvature of the graph using the proposed p-Laplacian with gradient terms for $p = 2$. The first image (on the top left) shows the initial contour, and, from top left to bottom right, the images demonstrate different steps of the contour evolution until convergence.*

image (in the sense of the mean color similarity). The main advantages of these graphs are, first, the efficiency of the approach as we work on a reduced version of the image and, second, the additional edges that allow us to connect unconnected objects in the natural image representation. The weighting function depends on the similarity between vertex colors (according to the image). The function $f_0 : V \to \mathbb{R}$ to be interpolated is initialized according to user-defined seeds (as presented in (6.11)). Figure 6 presents the initial image with user-defined seeds (red and green), the RAG, and the result of interpolation with the parameters $\alpha = \beta$, $p = 2$ (recovering the anisotropic Laplacian $\Delta_{w,2}$ (4.3)), and $w = s_1$.

Figure 5. *Illustration of interactive image segmentation using the proposed p-Laplacian with gradient terms for $p = 2$. The left image shows the original image image with superposed initial labels. The image on the right shows the result of the label diffusion.*

Figure 6. *Semisupervised image segmentation using a RAG. The segmentation is performed on a high-level graph: the RAG built from the initial image, extended with additional edges in order to connect similar but not adjacent regions. This construction enables us to diffuse labels through nonconnected objects (e.g., the flowers in the image).*

Real data clustering. In this paragraph we discuss label interpolation using (6.10) for real data clustering. The data is a set of 200 digits (zeros and ones) from the USPS database [32] that we want to cluster in two classes (zeros versus ones). To simplify the graph construction we consider each digit as a one-dimensional vector of 256 gray values, and the metric between the digits is a simple Euclidean distance. Then, the data are represented as a k-nn graph on which the clustering is processed for user-defined seeds (one per class). Figure 7 illustrates the graph with initial user-defined seeds and the resulting clustering, using $p = \infty$, $\alpha = \beta$ (recovering the ∞-Laplacian $\Delta_{w,\infty}$), and $w = s_1$.

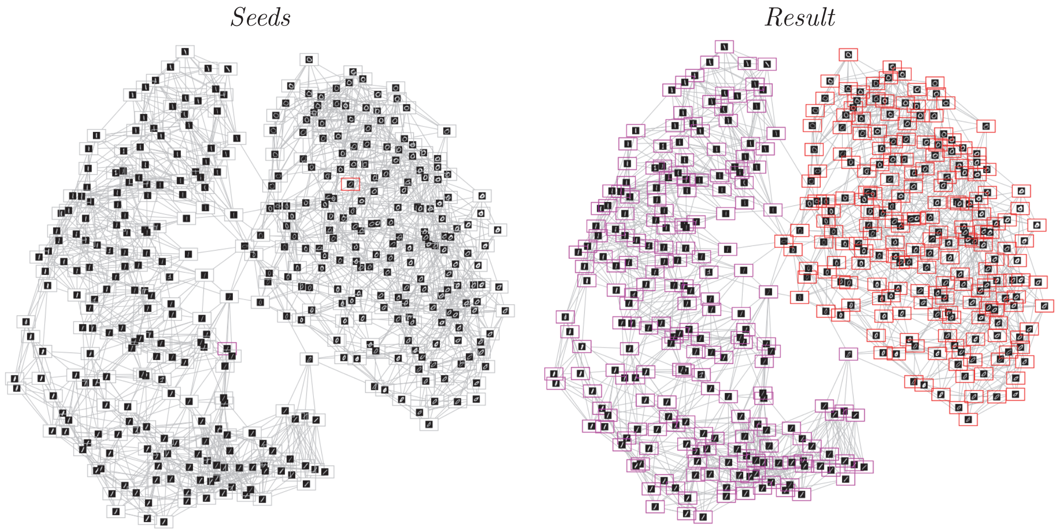

Figure 7. *Semisupervised data clustering for two classes on data picked from the USPS database (0's and 1's). The left image shows the graph with initial user-defined seeds, and the right image shows the result of the clustering.*

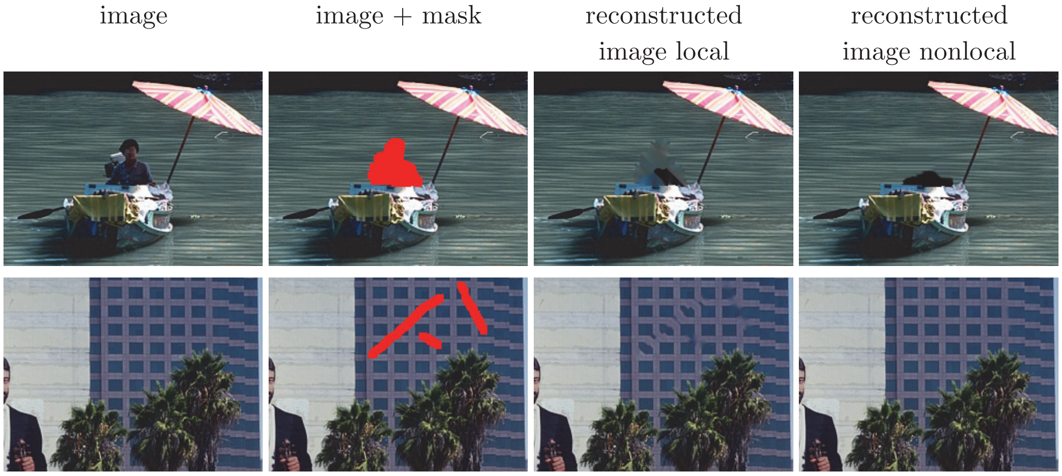

Figure 8. *Natural image inpainting in local and nonlocal configurations. The third column presents results with a local* 8*-adjacency graph. The fourth column presents nonlocal results using a* 31×31 *neighborhood window and* 15×15 *patches.*

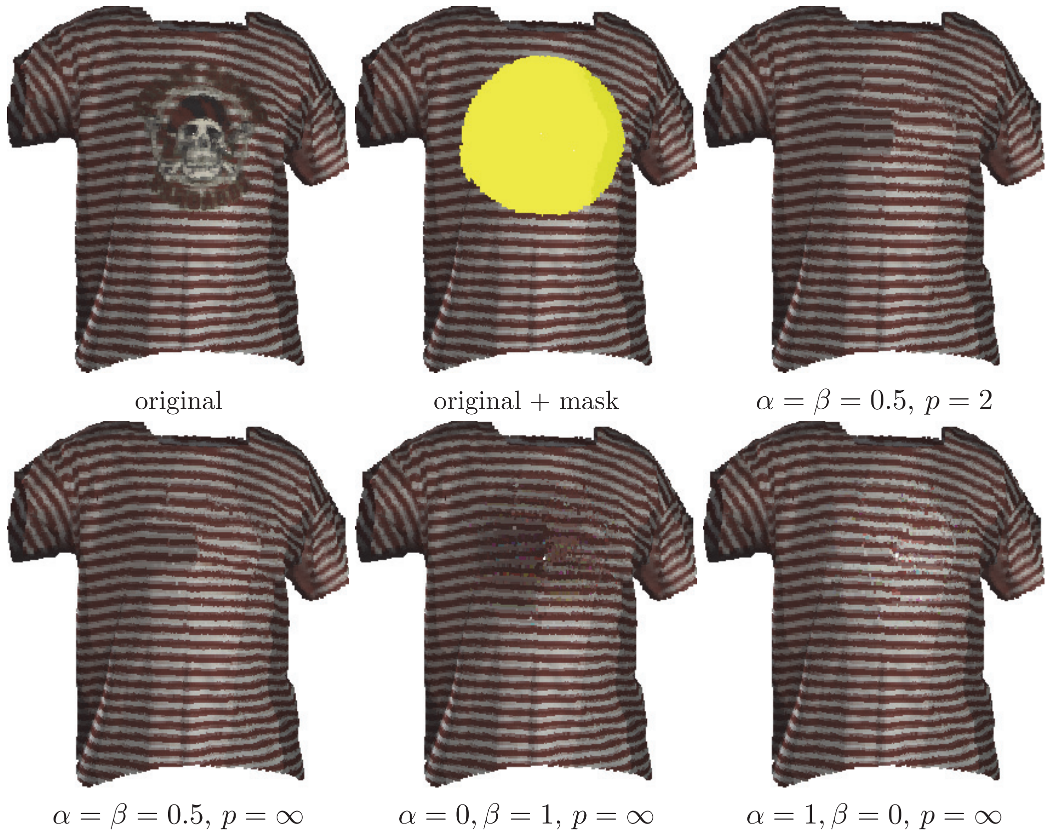

Figure 9. *Illustration of color inpainting on a 3D point cloud. Results are computed in the case $\alpha = \beta = 0.5$ for $p = 2$, and $\alpha = 0$, $\beta = 1$, $\alpha = 1$, $\beta = 0$ for $p = \infty$.*

6.3.2. Nonlocal image inpainting. Digital inpainting is a fundamental problem in image processing and has many applications in different fields. It can be simply summarized as reconstructing a damaged or incomplete image by filling the missing information in the incomplete regions. In recent years many methods have been developed for interpolating the geometry, the texture, or both geometry and texture. Among the interpolation methods that have been proposed, a number of methods are based on PDEs or variational methods; see [3, 49] and reference therein. Since the work of [12] on nonlocal filtering, many nonlocal methods for image inpainting have gained considerable attention in recent years. This is in part due to their superior performance in textured images, which represent a well-known weakness of purely local methods.

Recent works aim to unify local and nonlocal interpolation approaches [27]. A variational framework for nonlocal image inpainting has been presented in [3]. A discrete nonlocal regularization framework for image and manifold processing has been proposed in [26]. This framework has been used to present a unifying approach of local geometric methods and nonlocal exemplar-based ones for video inpainting. Considering (6.10) the inpainting problem can be formulated as follows: V_0 is the set of pixels with missing information, while $g : V \to \mathcal{H}(V)$ represents the known information and $f : V \to \mathcal{H}(V)$ is the image to be reconstructed. This is illustrated in Figure 8 on natural images with $\alpha = \beta = 0.5$ and $p = 2$ which corresponds to $\Delta_{w,2}$. The graph is either a local one with an 8-adjacency neighborhood (third column) or a nonlocal graph built using a 31×31 neighborhood window and 15×15 patches with patch similarity as a weighting function (fourth column).

Figure 9 illustrates results of graph inpainting for 3D point cloud color reconstruction. In this case, the graph is built as a nonlocal graph from the point cloud using the definition of patches on point cloud proposed by [36]. The function f to be interpolated associates a color vector to each vertex of the graph. Once the graph is built the problem is formulated in the same way as for the previous example of image inpainting. Results are presented for $\alpha = \beta = 0.5$, $p = 2$, corresponding to $\Delta_{w,2}$; $\alpha = \beta = 0.5$, $p = \infty$, corresponding to $\Delta_{w,\infty}$; $\alpha = 0$, $p = \infty$ corresponding to an erosion process; and $\alpha = 1$, $p = \infty$, corresponding to a dilation process.

7. Conclusion. In this paper, we have presented an overview of different versions of the Laplace operator in the continuous setting, i.e., the p-Laplacian and ∞-Laplacian and its related variants, the game p-Laplacian, and the nonlocal p-Laplacian. Subsequently we discussed how to translate these operators on graphs using our previous works on this topic. We then proposed a novel class of p-Laplacians and ∞-Laplacians on graphs, which can be expressed as a convex interpolation between two discrete upwind gradient terms, unifying our previous works on PdEs on graphs as a single operator. We have shown that an interesting feature of this representation is the fact that the respective steering parameters can be chosen data-dependent, which leads to adaptive filtering effects (nonlocal diffusion and morphological filtering) in different regions of the same data. We discussed the connection to local and nonlocal PDEs and a model from stochastic game theory known as the Tug-of-War game and showed that our proposed formulation, using $p = \infty$, leads to PdEs which coincide with value functions of different Tug-of-War games. We also applied this novel class of the p-Laplacian and ∞-Laplacian on two PdEs on weighted graphs, a parabolic equation, and an elliptic one using Dirichlet boundary conditions and proved important mathematical properties, e.g., existence and uniqueness of solutions. We have shown that the parabolic equation leads to a generalization of diffusion and mathematical morphology on graphs and that the elliptic PdEs with Dirichlet boundary conditions generalizes interpolation processes on discrete domains. Finally, we illustrated how this new class of p-Laplacians and ∞-Laplacians with gradient terms can be applied in many examples, i.e., segmentation, denoising, inpainting, and clustering. To underline the universal applicability of the proposed formulation we tested our algorithms on a wide range of data, i.e., classical images, triangulated meshes, and even unorganized data such as point clouds or databases.

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