

# Solving Minimal Surface Problems on Surfaces and Point Clouds

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**Abstract.** Minimal surface problems play an important role not only in physics or biology but also in mathematical signal and image processing. Although the computation of respective solutions is well-investigated in the setting of discrete images, only little attention has been paid to more complicated data, e.g., surfaces represented as meshes or point clouds. In this work we introduce a novel family of discrete total variation seminorms for weighted graphs based on the upwind gradient and incorporate them into an efficient minimization algorithm to perform total variation denoising on graphs. Furthermore, we demonstrate how to utilize the latter algorithm to uniquely solve minimal surface problems on graphs. To show the universal applicability of this approach, we illustrate results from filtering and segmentation of 3D point cloud data.

**Key words:** Minimal surface problem, discrete total variation, upwind differences, graphs, segmentation, denoising, point cloud data

## 1 Introduction

Variational models and partial differential equation-based (PDE) methods are a fundamental tool in mathematical image processing and computer vision. The respective mathematical theory is well-understood and allows for important conclusions, e.g., existence and uniqueness of solutions. Traditionally, algorithms from this field are investigated and applied in the domain of Euclidean spaces. However, in many physical and biological contexts there exist problems involving PDEs on more complex and irregular domains. In particular, due to the technical advances in producing affordable 3D sensors in the past few years more and more applications are related to data defined on polygonal surfaces and point clouds. In general, the study of variational methods and PDEs needs significantly more effort in these cases. Moreover, in the case of point clouds the data is directly sampled from the underlying surface. Since there is a-priori no connectivity of these points provided this leads to additional problems in processing these data.

Today, variational problems and PDEs on surfaces can be tackled by one of the following techniques. First, there are methods which approximate surfaces by

polygonal meshes and locally parametrize them by suitable functions, e.g., finite element methods [10]. Another possibility is to implicitly represent surfaces as level sets of a Lipschitz continuous function on Euclidean domains and use level set methods to solve PDEs [2]. Another example is the closest point method proposed in [16], which replaces surface differentials with Cartesian differentials of its closest point extension. Recently, discrete differential geometry on surfaces [3, 13] gained increased attention, especially in computer graphics. While most of these methods do not immediately extend to arbitrary point cloud data, they could generally be used if these data were reconstructed into suitable surface representations - a process that can be very complex and cost-intensive in terms of computational effort. For a discussion of these different techniques with their advantages and disadvantages, see e.g., [13, 14].

In this work we focus on the well-known minimal surface problem, see e.g., [9, 18] and references therein. Deviating from the techniques discussed above, we perform discrete calculus on surfaces using graph-based representations [1, 11, 12, 14]. Based on our previous work in [14] we represent the given data as a graph and use the proposed framework in [11] to transfer differential operators to graphs. We are able to solve minimal surface problems for meshes and arbitrary point clouds in a unified manner using this relatively simple approach.

### 1.1 Motivation

The minimal surface problem [5, 8] consists of finding a subset  $\Sigma$  of an open, bounded domain  $\Omega \subset \mathbb{R}^n$ , which partitions  $\Omega$  according to some external energy  $g: \Omega \rightarrow \mathbb{R}$  while having a minimal surface  $\Gamma \subset \Omega$  with respect to an adequate measure. Denoting this measure of the surface of  $\Sigma$  in  $\Omega$  by  $\text{Per}(\Sigma; \Omega)$  one can formulate the minimal surface problem as follows:

For a fixed  $\lambda > 0$  and  $z \in \mathbb{R}$  find a minimizing set  $\Sigma \subset \Omega$  of the energy

$$E(\Sigma) = \text{Per}(\Sigma; \Omega) + \lambda \int_{\Sigma} z - g(x) \, dx . \quad (1)$$

Here,  $\lambda$  is a regularization parameter controlling the smoothness of  $\Gamma$ . Solutions of the problem (1) are not unique as the problem is non-convex [8].

There exist various tasks in signal processing which can be associated to the minimal surface problem. To the best of our knowledge a unified manner for computing local and nonlocal minimal surface solutions on graphs has not been investigated so far. To motivate our work we discuss three examples from the literature. First, given an image  $f: \Omega \rightarrow \mathbb{R}$  and setting the external energy to  $g(x) = f(x)$ , then solving (1) will perform a segmentation of  $f$  according to a threshold induced by  $z \in \mathbb{R}$ . Decreasing the parameter  $\lambda$  enforces regularity of the segmentation surface and thus makes the segmentation more robust to noise and outliers, e.g., see [18]. A second example is given by the popular Chan-Vese segmentation method in [8]. Based on an initial guess of a partition  $\Sigma \subset \Omega$ , the authors compute two mean values  $c_1, c_2 \in \mathbb{R}$  of the image regions of  $f$  induced by  $\Sigma$ . Subsequently, the Chan-Vese method performs the actual segmentation step

by setting  $z = 0$  and  $g(x) = (f(x) - c_1)^2 - (f(x) - c_2)^2$  and solving (1). In fact, this clusters the image values according to the estimated cluster centers  $c_1$  and  $c_2$ . Another problem is the computation of the mean curvature flow. As the authors in [5] discuss, given an initial surface  $\Gamma$  of  $\Sigma$  and setting  $z = 0$  and  $g(x) = d_\Gamma(x)$  as the signed Euclidean distance of a point  $x$  to  $\Gamma$ , one approximates the mean curvature flow of  $\Gamma$  by computing a sequence of solutions to (1). This is especially interesting for modeling surface evolution in physics or biology.

## 1.2 Contributions

Our aim in this paper is to establish a unified method to solve various important tasks from image and data processing which can be formulated as minimal surface problems on data with arbitrary topology, such as point clouds. First, based on the notion of discrete total variation on graphs we introduce a new family of total variation seminorms based on the upwind gradient operator. We give two important relationships, namely the discrete coarea formula for graphs and a link between minimal surface problems and the popular ROF denoising model. Finally, we give an efficient iterative scheme to uniquely solve minimal surface problems on graphs. The advantage of this approach is that it is universally applicable for various tasks, such as total variation denoising, segmentation, data clustering, or computation of the mean curvature flow.

## 2 Methods

We start with the basic notation and mathematical basics of finite weighted graphs and give the definition of weighted finite differences for the latter. Subsequently, we formulate the notion of the discrete total variation of a function defined on graph vertices in Section 2.1 and introduce a novel family of discrete total variation seminorms based on the upwind gradient operator. For certain special cases of these seminorms we are able to prove a proposition in Section 2.2 which corresponds to the coarea formula in the continuous setting. Based on this, we give an important relationship on weighted graphs between total variation denoising problems and the minimal surface problem.

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$  be a weighted graph without loops and self-referencing, for which  $w: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}^+$  is a weighting function depending on the interactions of the vertices in the finite set  $\mathcal{V} = \{x_1, \dots, x_N\}$  given by the edge set  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ . In the following we assume that the weighting function  $w$  is symmetric and hence  $\mathcal{G}$  is an undirected graph. We denote by  $\mathcal{H}(\mathcal{V})$  the Hilbert space of real-valued functions on the vertices of the graph and by  $\mathcal{H}(\mathcal{E})$  the Hilbert space of real-valued functions on the graph edges. For  $f \in \mathcal{H}(\mathcal{V})$  we define the  $p$ -norm as:

$$\|f\|_p = \left( \sum_{x_i \in \mathcal{V}} |f(x_i)|^p \right)^{1/p}, \quad 1 \leq p < \infty \quad \text{and} \quad \|f\|_\infty = \max_{x_i \in \mathcal{V}} |f(x_i)|. \quad (2)$$

Following [11], we introduce weighted finite differences  $d_w: \mathcal{H}(\mathcal{V}) \rightarrow \mathcal{H}(\mathcal{E})$  by:

$$(d_w f)(x_i, x_j) = \sqrt{w(x_i, x_j)}(f(x_j) - f(x_i)) , \quad (3)$$

and for  $H \in \mathcal{H}(\mathcal{E})$  one is able to give the adjoint operator  $d_w^*: \mathcal{H}(\mathcal{E}) \rightarrow \mathcal{H}(\mathcal{V})$  as,

$$(d_w^* H)(x_i) = \sum_{x_j \sim x_i} \sqrt{w(x_i, x_j)}(H(x_j, x_i) - H(x_i, x_j)) . \quad (4)$$

We define a weighted gradient operator  $\nabla_w$  of a function  $f \in \mathcal{H}(\mathcal{V})$  as:

$$(\nabla_w f)(x_i) = ((d_w f)(x_i, x_j))_{x_j \in \mathcal{V}}^T , \quad (5)$$

Based on the difference operator in (3) one can additionally get weighted upwind differences similar to upwind discretization schemes, e.g., see [14], by:

$$\begin{aligned} (d_w^+ f)(x_i, x_j) &= \max(0, (d_w f)(x_i, x_j)) , \\ (d_w^- f)(x_i, x_j) &= \min(0, (d_w f)(x_i, x_j)) = -\max(0, (d_w f)(x_j, x_i)) . \end{aligned} \quad (6)$$

Based on these weighted upwind difference one gets the discrete upwind weighted gradient operators  $\nabla_w^+$  and  $\nabla_w^-$  analogously to (5).

## 2.1 Discrete total variation for graphs

Based on the gradient operator of a function  $f \in \mathcal{H}(\mathcal{V})$  introduced in (5) one can introduce a measure of the discrete total variation (TV) of  $f$  on a weighted graph  $\mathcal{G}$ . For this we define a family of discrete total variation seminorms as:

$$\begin{aligned} \|f\|_{TV,p} &= \sum_{x_i \in \mathcal{V}} \|(\nabla_w f)(x_i)\|_p = \sum_{x_i \in \mathcal{V}} \left( \sum_{x_j \sim x_i} |(d_w f)(x_i, x_j)|^p \right)^{1/p}, \quad 1 \leq p < \infty, \\ \|f\|_{TV,\infty} &= \sum_{x_i \in \mathcal{V}} \|(\nabla_w f)(x_i)\|_\infty = \sum_{x_i \in \mathcal{V}} \max_{x_j \sim x_i} |(d_w f)(x_i, x_j)|, \quad p = \infty. \end{aligned} \quad (7)$$

Inspired by the definition of a discrete upwind total variation seminorm by the authors in [6], we introduce a novel family of TV seminorms on graphs based on the weighted upwind differences in (6),

$$\begin{aligned} \|f\|_{TV,p}^+ &= \sum_{x_i \in \mathcal{V}} \|(\nabla_w^+ f)(x_i)\|_p = \sum_{x_i \in \mathcal{V}} \left( \sum_{x_j \sim x_i} |(d_w^+ f)(x_i, x_j)|^p \right)^{1/p} \\ &= \sum_{x_i \in \mathcal{V}} \left( \sum_{x_j \sim x_i} (w(x_i, x_j))^{p/2} |\max(0, f(x_j) - f(x_i))|^p \right)^{1/p}, \quad 1 \leq p < \infty, \\ \|f\|_{TV,\infty}^+ &= \sum_{x_i \in \mathcal{V}} \|(\nabla_w^+ f)(x_i)\|_\infty = \sum_{x_i \in \mathcal{V}} \max_{x_j \sim x_i} |(d_w^+ f)(x_i, x_j)| \\ &= \sum_{x_i \in \mathcal{V}} \max_{x_j \sim x_i} \left( \sqrt{w(x_i, x_j)} |\max(0, f(x_j) - f(x_i))| \right), \quad p = \infty. \end{aligned} \quad (8)$$

Using the definition of the weighted upwind differences  $d_w^-$  in (6), we introduce analogously a novel family of TV seminorms  $\|\cdot\|_{TV,p}^-$  and  $\|\cdot\|_{TV,\infty}^-$ . Finally, we are also interested in a discrete total variation seminorm of the form:

$$\|f\|_{TV,\infty}^\pm = \frac{1}{2} \sum_{x_i \in \mathcal{V}} \|(\nabla_w^+ f)(x_i)\|_\infty + \|(\nabla_w^- f)(x_i)\|_\infty. \quad (9)$$

Given a subset of vertices  $\mathcal{A} \subset \mathcal{V}$  of a weighted graph  $\mathcal{G}$ , we define a family of perimeters of  $\mathcal{A}$  as the total variation (see Section 2.1) of its respective indicator function  $\chi_{\mathcal{A}}$  for  $1 \leq p < \infty$ :

$$\begin{aligned} \text{Per}_p(\mathcal{A}; \mathcal{V}) &= \frac{1}{p} \|\chi_{\mathcal{A}}\|_{TV,p} = \frac{1}{p} \sum_{x_i \in \mathcal{V}} \|(\nabla_w \chi_{\mathcal{A}})(x_i)\|_p, \quad 1 \leq p < \infty, \\ \text{Per}_\infty(\mathcal{A}; \mathcal{V}) &= \|\chi_{\mathcal{A}}\|_{TV,\infty} = \sum_{x_i \in \mathcal{V}} \|(\nabla_w \chi_{\mathcal{A}})(x_i)\|_\infty, \quad p = \infty. \end{aligned} \quad (10)$$

For the novel family of total variation seminorms on graphs in (8) we can analogously introduce families of perimeters as:  $\text{Per}_\infty^+(\mathcal{A}; \mathcal{V}) = \frac{1}{2p} \|\chi_{\mathcal{A}}\|_{TV,\infty}^+$ ,  $\text{Per}_\infty^-(\mathcal{A}; \mathcal{V}) = \frac{1}{2p} \|\chi_{\mathcal{A}}\|_{TV,\infty}^-$ , and for (9) we obtain  $\text{Per}_\infty^\pm(\mathcal{A}; \mathcal{V}) = \|\chi_{\mathcal{A}}\|_{TV,\infty}^\pm$ .

## 2.2 The coarea formula and total variation denoising

Based on the introduced families of perimeters of subsets of vertices above we can formulate the minimal surface problem (1) for a weighted graph  $\mathcal{G}$  as follows:

For a fixed  $\lambda > 0$  and  $z \in \mathbb{R}$  find a minimizing set of vertices  $\mathcal{A} \subset \mathcal{V}$  of

$$E(\mathcal{A}) = \text{Per}(\mathcal{A}; \mathcal{V}) + \lambda \sum_{x_i \in \mathcal{A}} z - g(x_i), \quad (11)$$

for which  $\text{Per}(\mathcal{A}; \mathcal{V})$  is one of the perimeters introduced in Section 2.1. Equivalently, the minimization problem (11) can be reformulated in terms of the characteristic function  $\chi_{\mathcal{A}}$  of the subset  $\mathcal{A}$  as:

For a fixed  $\lambda > 0$  and  $z \in \mathbb{R}$  find an optimal function  $\chi_{\mathcal{A}} \in \mathcal{I}(\mathcal{V})$  of

$$E(\chi_{\mathcal{A}}) = \|\chi_{\mathcal{A}}\|_{TV} + \lambda \sum_{x_i \in \mathcal{V}} \chi_{\mathcal{A}}(x_i)(z - g(x_i)), \quad (12)$$

for which  $\|\cdot\|_{TV}$  is one of the total variation seminorms on graphs introduced in Section 2.1 and  $\mathcal{I}(\mathcal{V}) \subset \mathcal{H}(\mathcal{V})$  is the subset of indicator functions, which only take the values 0 and 1. As gets clear, the minimization problem (12) (and thus also (11)) is nonconvex, as the admissible set of functions in  $\mathcal{I}(\mathcal{V})$  is nonconvex [8]. There exist different approaches to solve the discrete minimal surface problem. One popular approach is known as exact convex relaxation and performs optimization on a subset of functions which are bounded by the interval  $[0, 1]$  and subsequently perform thresholding, see [8] for details.

However, we refrain from using this approach for two reasons. First, we want to compute minimal surface solutions on graphs which can be used for different important image processing tasks, such as filtering and segmentation. Thus, we need a framework which covers these applications in a unified manner, which is not possible with the exact convex relaxation approach discussed above. Second, the latter method gives no conclusions about uniqueness of solutions, which is a desirable property in image and data processing.

In the case of images the approach in [4] exploits a useful relationship between minimal surface problems and the well-known Rudin-Osher-Fatemi (ROF) total variation denoising problem in [17]. This relationship is based on the coarea formula. Thus, we formulate the discrete coarea formula on graphs for certain families of the discrete TV seminorms introduced in Section 2.1. Note that these relationships can also be found in [1].

**Proposition 1 (Coarea formula for graphs)**

Let  $u \in \mathcal{H}(\mathcal{V})$  be a vertex function of a weighted graph  $\mathcal{G}$ . Then in cases of the discrete TV seminorms given by  $p = 1$  and  $*$   $\in \{ \cdot, +, - \}$  as in (9) the coarea formula for graphs holds:

$$\|u\|_{TV,1}^* = \int_{-\infty}^{\infty} \|\chi_{\{u>t\}}\|_{TV,1}^* dt = \int_{-\infty}^{\infty} \text{Per}_1^* (\{x_i \in \mathcal{V} : u(x_i) > t\}; \mathcal{V}) dt . \quad (13)$$

*Proof.* The proof is based on the definition of the discrete TV seminorms in Section 2.1 and the two identities  $a - b = \int_{-\infty}^{\infty} \chi_{\{b>t\}} - \chi_{\{a>t\}} dt$  and  $\max(0, u(x_j) - u(x_i)) = -\min(0, u(x_i) - u(x_j))$  as given in (6) for vertices  $x_i, x_j \in \mathcal{V}$ .

**Remark 1** In the case of unweighted graphs, i.e.,  $w \equiv c \in \mathbb{R}$ , the coarea formula (13) holds true for the discrete TV seminorm  $\|\cdot\|_{TV,\infty}$ , and also for the total variation seminorm  $\|\cdot\|_{TV,\infty}^\pm$  in (9). This is of particular interest in the special case of traditional image processing with the weighting function  $w \equiv \frac{1}{h^2}$ .

Following [4, Prop. 2.7] one can derive a useful relationship between the discrete minimal surface problem on graphs (12) and the ROF denoising problem.

**Proposition 2 (Solving minimal surface problems by TV denoising)**

Let  $\mathcal{G}$  be a graph,  $\lambda > 0$  a fixed parameter,  $g \in \mathcal{H}(\mathcal{V})$ , and  $\hat{u} \in \mathcal{H}(\mathcal{V})$  the unique solution of the discrete ROF total variation denoising functional,

$$\min_{u \in \mathcal{H}(\mathcal{V})} \frac{\lambda}{2} \|u - g\|_2^2 + \|u\|_{TV} , \quad (14)$$

for which  $\|\cdot\|_{TV}$  is one of the introduced discrete TV seminorms in Section 2.1 that fulfill the coarea formula in Proposition 1.

Then, for almost every  $z \in \mathbb{R}$ , the indicator function  $\hat{\chi} \in \mathcal{H}(\mathcal{V})$  with

$$\hat{\chi}(x) = \begin{cases} 1 , & \text{if } \hat{u}(x) > z , \\ 0 , & \text{else ,} \end{cases} \quad (15)$$

is a solution of the discrete minimal surface problem (12). In particular, for all  $z$  but a countable set, the solution is even unique.

*Proof.* The proof of this proposition is based on the coarea formula in Proposition 1 and follows directly the proof in [4, Prop. 2.7].

In summary Proposition 2 states that one can obtain a unique solution of the discrete minimal surface problem by solving an associated strictly convex TV denoising problem and perform a single thresholding step afterwards. Hence, this basically reduces the solution of the minimization problem (12) (and thus also of (11)) to solving the ROF model (14) on graphs. Note that by this one can solve various important problems in image and data processing not only in a unified manner, but also very efficiently. We discuss this in the following section.

### 3 Algorithm

As discussed in Section 2 we can uniquely solve many important problems from data and image processing in a unified manner. For this we simply have to efficiently minimize an associated TV denoising energy on graphs. As the ROF problem in (14) is well-studied in the literature there exist various ways how to solve it. In this work we use the Chambolle-Pock (CP) primal-dual method proposed in [7] to minimize (14) since it connects different efficient minimization algorithms from the literature. Let  $U, V$  be finite-dimensional real vector spaces equipped with an inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$ . The CP method is designed to minimize convex problems of the form:

$$\min_{u \in U} D(u) + R(Ku) , \quad (16)$$

where in this setting  $K: U \rightarrow V$  is a linear operator and  $D, R$  are convex, lower-semicontinuous functions. The problem in (16) is associated with a saddle-point problem given by:

$$\min_{u \in U} \max_{v \in V} \langle Ku, v \rangle + D(u) - R^*(v) , \quad (17)$$

where  $R^*$  is the conjugate of  $R$ . To solve the saddle-point problem (17) the authors propose the following general iterative minimization scheme:

$$v^{n+1} = \text{prox}_{\sigma R^*}(v^n + \sigma K \bar{u}^n) \quad (18a)$$

$$u^{n+1} = \text{prox}_{\tau D}(u^n - \tau K^* v^{n+1}) \quad (18b)$$

$$\bar{u}^{n+1} = u^{n+1} + \theta(u^{n+1} - u^n) \quad (18c)$$

In this context  $K^*$  denotes the adjoint operator of  $K$ ,  $\text{prox}$  is the proximal operator with:

$$\text{prox}_f(x) = \operatorname{argmin}_{y \in X} \|y - x\|_X^2 + f(y)$$

and  $\sigma, \tau, \theta > 0$  are step size parameters. The convergence of the CP algorithm in (18) is guaranteed for  $\theta = 1$  and  $\sigma\tau\|K\|^2 < 1$ . Since the ROF problem is strictly convex, using a modified version of the CP algorithm in [7] yields a convergence rate of  $\mathcal{O}(1/N^2)$ .

One is able to use the primal-dual CP algorithm on graphs in the setting of the minimization problem (14) if one chooses:  $K = \nabla$ ,  $D = \lambda \|\cdot - f\|_2^2$ , and  $R = \|\cdot\|_1$ . In this setting it is reasonable to use the dual formulation of the discrete TV seminorms introduced in Section 2.1:

$$\|u\|_{TV} = \max_{\substack{v \in \mathcal{D}(\mathcal{V}) \\ \|v\|_\infty \leq 1}} \sum_{x_i \in \mathcal{V}} \nabla_w u(x_i) v(x_i) = \max_{\substack{v \in \mathcal{D}(\mathcal{V}) \\ \|v\|_\infty \leq 1}} - \sum_{x_i \in \mathcal{V}} u(x_i) \operatorname{div}_w v(x_i), \quad (19)$$

for which  $\mathcal{D}(\mathcal{E})$  is the Hilbert space of vertex functions  $v: \mathcal{V} \rightarrow \mathbb{R}^N$  and  $\operatorname{div}_w$  is the divergence operator given by the negative adjoint operator  $-d_w^*$  in (4). Note that the condition  $\|v\|_\infty = \max_{x_i \in \mathcal{V}} \|v(x_i)\|_q \leq 1$  is based on the dual norm of the inner norm  $\|\cdot\|_p$  in (2) which fulfills the Hölder conjugate condition, i.e.,  $\frac{1}{p} + \frac{1}{q} = 1, 1 < p, q < \infty$  for which  $q = \infty$  is the Hölder conjugate of  $p = 1$  and vice versa. Following [7], one is able to show that the conjugate of a norm can be represented as the indicator function of its dual norm unit ball, i.e.,  $\|\cdot\|^* = \chi_{\|\cdot\|_* \leq 1}$ . Furthermore, the proximal operator of a indicator function of a convex set is the respective projection onto this set, i.e.,  $\operatorname{prox}_{\chi_C}(x) = \operatorname{proj}_C(x)$ .

In case of the discrete total variation seminorms in (7) and vertex functions  $u, \bar{u} \in \mathcal{H}(\mathcal{V})$  and  $v \in \mathcal{D}(\mathcal{V})$  the application of the CP primal-dual algorithm for the problem (14) thus leads to the following minimization scheme on graphs:

$$v^{n+1} = \frac{v^n + \sigma \nabla_w \bar{u}^n}{\max(1, \|v^n + \sigma \nabla_w \bar{u}^n\|_q)} \quad (20a)$$

$$u^{n+1} = \frac{u^n + \tau (\operatorname{div}_w v^{n+1} + \lambda f)}{\lambda \tau + 1} \quad (20b)$$

$$\bar{u}^{n+1} = u^{n+1} + \theta (u^{n+1} - u^n) \quad (20c)$$

with the initialization  $u^0 = \bar{u}^0 = f$  and  $v^0 \equiv 0$ .

In case of the novel discrete upwind total variation seminorms in (8) the CP algorithm is not directly applicable, since the upwind gradient operators  $\nabla_w^+$  and  $\nabla_w^-$  are not linear. Following [6], one can deal with this problem by adding an additional constraint to the dual formulation of the discrete upwind total variation seminorms as follows:

$$\|u\|_{TV}^+ = \max_{\substack{v \in \mathcal{D}(\mathcal{V}) \\ \|v\|_\infty \leq 1}} \sum_{x_i \in \mathcal{V}} \nabla_w^+ u(x_i) v(x_i) = \max_{\substack{v \in \mathcal{D}(\mathcal{V}) \\ \|v\|_\infty \leq 1 \\ v_i \geq 0}} - \sum_{x_i \in \mathcal{V}} u(x_i) \operatorname{div}_w v(x_i), \quad (21)$$

and analogously the condition  $v_i \leq 0$  for the discrete upwind TV seminorm  $\|\cdot\|_{TV}^-$ . By this the convex set of possible minimizers of the dual minimization step gets restricted by an additional constraint and thus we have to adapt the projection step in (20a) of the deduced minimization scheme:

$$v^{n+1} = \frac{\max(0, v^n + \sigma \nabla_w \bar{u}^n)}{\max(1, \|\max(0, v^n + \sigma \nabla_w \bar{u}^n)\|_q)},$$

for the case of the discrete upwind TV seminorm  $\|\cdot\|_{TV}^+$  and for the case of  $\|\cdot\|_{TV}^-$  one derives:

$$v^{n+1} = \frac{\min(0, v^n + \sigma \nabla_w \bar{u}^n)}{\max(1, \|\min(0, v^n + \sigma \nabla_w \bar{u}^n)\|_q)}.$$

Finally, in the case of the discrete total variation seminorm  $\|\cdot\|_{TV,\infty}^\pm$  in (9) one has the following projection step:

$$v^{n+1} = \frac{\max(0, v^n + \sigma \nabla_w \bar{u}^n) + \min(0, v^n + \sigma \nabla_w \bar{u}^n)}{\max(2, \|\max(0, v^n + \sigma \nabla_w \bar{u}^n)\|_1 + \|\min(0, v^n + \sigma \nabla_w \bar{u}^n)\|_1)}.$$

The alternating minimization scheme (20a) is terminated after the relative change of the primal variable  $u$  falls under a certain accuracy, e.g.,  $\epsilon < 10^{-5}$ . To finally obtain a solution of the minimal surface problem (12) one simply computes the indicator function  $\chi_{\mathcal{A}} = 1$  for  $\hat{u}(x_i) > z$  for all  $x_i \in \mathcal{V}$  and 0 else. Note that the computed solution  $\hat{u}$  in fact gives a whole family of solutions of (12) since its respective level sets are minimal surface solutions for the respective value of  $z$  [5]. This is in particular interesting, as it allows to adapt the computed solution in real time, e.g., by changing the threshold value with a slider [18].

## 4 Results

In the following we demonstrate the universal applicability of our approach by illustrating applications from filtering and segmentation of arbitrary point cloud data. It gets clear that the proposed algorithm in Section 3 is applicable for any weighted graph and hence for a huge range of possible data, e.g., for images and meshes [11, 14]. For the sake of brevity we discuss only the most complex case of arbitrary 3D point cloud data, which we took from [15]. We connect the neighborhoods of vertices using a symmetric k-nearest-neighbour (kNN) approach as described in [14]. Subsequently, we estimate the normal of the induced surface (if it is not given) in each vertex. We utilize this normal to determine an unique orientation of a patch within the tangent plane. By this we are able to average the signal values of the respective vertices which are projected into the patch cells. We compute the weight function of our graph based on the patch distance.

### 4.1 Filtering

For color filtering 3D point cloud data we simply solve the ROF denoising problem (14) for each color channel and thus filter the colors of the 3D points. In Figure 1 we projected a synthetic texture showing a strip pattern onto a 3D scanned vase. Subsequently, we added for each color channel Gaussian noise with mean  $\mu = 0$  and standard deviation  $\sigma = 40$  as shown in Figure 1a. We keep the regularization parameter fixed as  $\lambda = \frac{1}{60}$  and use the isotropic ( $p = 2$ ) discrete TV seminorm (7) to filter each color channel of the data. In this experiment we illustrate the impact of the graph construction by building a local

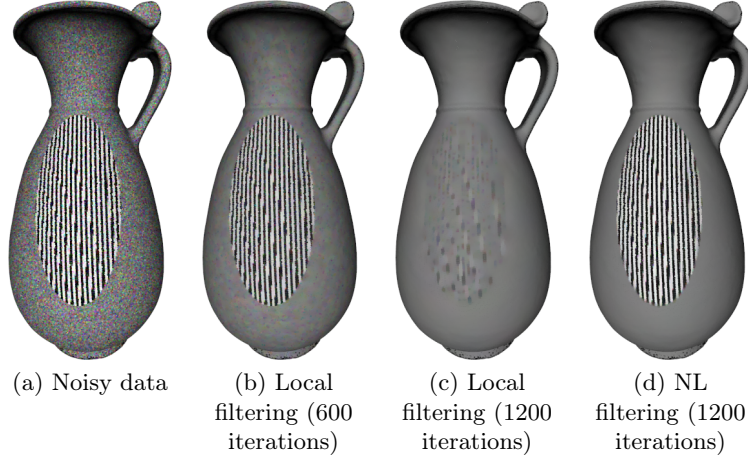


Fig. 1: Results of local and nonlocal (NL) color filtering on a 3D point cloud

graph with a constant weight function and a nonlocal graph with a high value of  $k$  and a weight function based on the patch distance as described in [14]. Figures 1b and 1c show the result of local filtering the point cloud data for 600 and 1200 iterations, respectively. Clearly, it is unavoidable to blur the texture in order to suppress the impact of noise on the whole data. However, constructing a nonlocal graph enables us to remove the noise from the background while preserving the texture. The second application of the proposed algorithm is geometric filtering of 3D point cloud data. Figure 2a shows a scanned human face with lots of misplaced points, which can be interpreted as geometric noise. We use the same parameters as above but instead of the color information of each vertex we filter their geometric coordinates as described in [14]. We compare the results of TV denoising using the normal and the newly proposed family of discrete TV seminorms based on the upwind gradient in (8) in Figures 2b and 2c, respectively. Although the effect is only small, one can see that the upwind gradient gives smoother surfaces without losing too much details of the face.

## 4.2 Segmentation

Finally, we perform segmentation of 3D point cloud data based on different features by using the Chan-Vese segmentation approach in [8] as discussed in Section 1.1. On the first data set of a scanned vase in Figure 3a we add several elliptic forms with a strip texture. We use the local standard deviation within the estimated patches as a scalar descriptor to perform binary segmentation of the surface. By using the grayscale values of the vertices directly we would not get the wanted segmentation result, as the background intensity lies between the grayscale values of the texture. In Figure 3b we show results for a mug surface with an color image of flowers after binary segmentation of the red color channel.

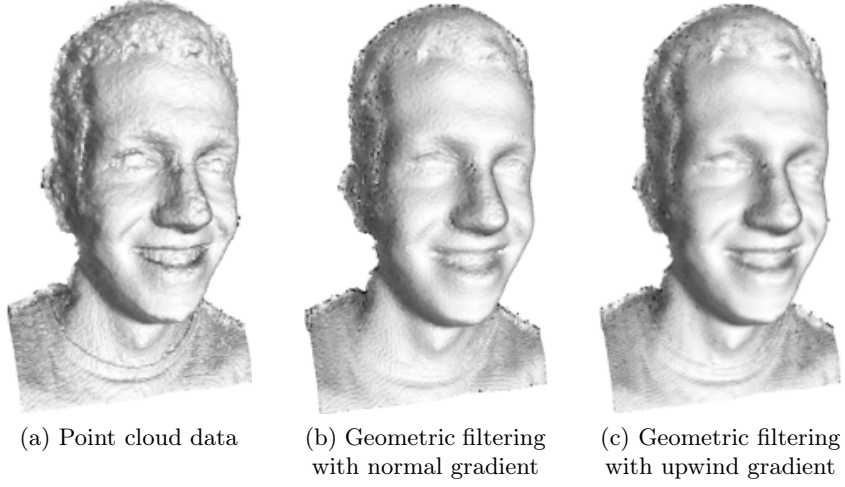


Fig. 2: Geometric filtering on 3D point cloud data

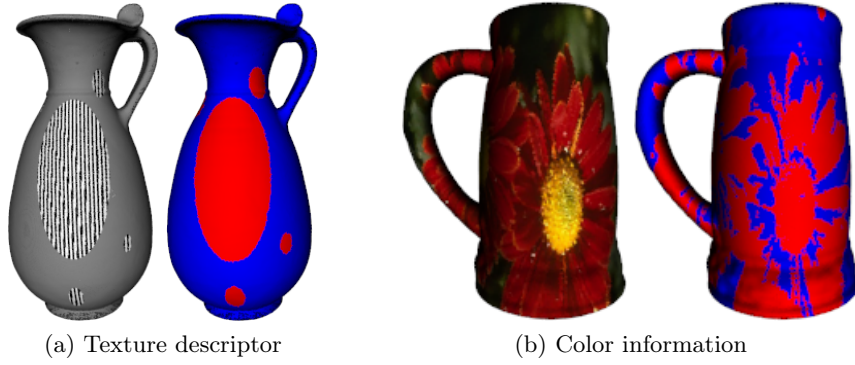


Fig. 3: Segmentation of two 3D point clouds using different features

## 5 Discussion

In this work we investigated an approach of solving minimal surface problems for arbitrary surfaces such as meshes and point cloud data via an relatively simple approach using graph-based representations. By approximating differential operators by weighted finite differences we were able to formulate the minimal surface problem on graphs. Furthermore, we introduced a novel family of discrete total variation seminorms based on the upwind gradient in this context. To uniquely solve minimal surface problems on graphs we transferred the Chambolle-Pock minimization algorithm to this domain. The importance and universal applicability of this approach is illustrated by applications from filtering and segmentation of point cloud data. In future work we would like to extend the range of applications to data clustering and mean curvature flow on graphs.

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